

Iterative Processes

Worked Solutions • Tables of Values • n th Terms

Section A Iterative Process $\longrightarrow$ Table of Values

A1. Consider the iterative process

$$U_{n+1} = U_n + 7, \quad U_1 = 4$$

(a) The completed table is:

n	1	2	3	4	5	6
U_n	4	11	18	25	32	39

(b) **Solution.** Each term is 7 more than the previous term. The sequence starts at 4 and increases by 7 each time. In symbols, $U_n = 7n - 3$.

A2. Consider the iterative process

$$V_{k+1} = 3V_k, \quad V_1 = 2$$

(a) The completed table is:

k	1	2	3	4	5
V_k	2	6	18	54	162

(b) **Solution.** Each term is 3 times the previous term. This is a geometric sequence with starting value 2 and common ratio 3.

A3. Consider the iterative process

$$F_{j+1} = F_j - 9, \quad F_0 = 72$$

(a) The completed table is:

j	0	1	2	3	4	5	6
F_j	72	63	54	45	36	27	18

(b) **Solution.** Continuing the pattern:

$$F_7 = 9, \quad F_8 = 0, \quad F_9 = -9.$$

Since $F_8 = 0$ is not negative, the first negative term is $F_9 = -9$. Therefore the first negative value occurs at $j = 9$.

A4. Consider the iterative process

$$P_{n+1} = \frac{P_n}{4}, \quad P_1 = 1024$$

(a) The completed table is:

n	1	2	3	4	5
P_n	1024	256	64	16	4

(b) Writing each value as a power of 4:

n	1	2	3	4	5
P_n	4^5	4^4	4^3	4^2	4^1

(c) From the pattern, the exponent decreases by 1 each time:

$$P_n = 4^{6-n}.$$

(d) Using the laws of indices:

$$P_n = 4^{6-n} = 4^6 \cdot 4^{-n} = 4096 \left(\frac{1}{4}\right)^n.$$

Section B Different representations of sequences

The completed rows are shown below.

B1.

Iterative Process	Table (first 5 terms)	n th Term Formula
$U_{n+1} = U_n + 3, \quad U_1 = 5$	5, 8, 11, 14, 17, ...	$U_n = 3n + 2$

B2.

Iterative Process	Table (first 5 terms)	n th Term Formula
$F_{j+1} = \frac{1}{2}F_j, \quad F_1 = 80$	80, 40, 20, 10, 5, ...	$F_j = 80 \left(\frac{1}{2}\right)^{j-1}$

B3. A quantity decreases by 15% each step, starting at $V_0 = 200$.

Iterative Process	Table ($n = 0$ to 4, 2 d.p.)	n th Term Formula
$V_{n+1} = 0.85V_n, \quad V_0 = 200$	200.00, 170.00, 144.50, 122.83, 104.40	$V_n = 200(0.85)^n$

Solution note. A decrease of 15% means the quantity keeps 85%, so it is multiplied by 0.85 at each step. Since the starting value is V_0 , the formula uses the power n , not $n - 1$.

B4.

Iterative Process	Table (first 5 terms, 2 d.p.)	n th Term Formula
$Q_{n+1} = \frac{Q_n}{2} + 6, \quad Q_1 = 100$	100.00, 56.00, 34.00, 23.00, 17.50	No simple formula from the methods above; see below.

What is different about this question?

Solution. This recurrence is neither arithmetic nor geometric. It combines two operations: halve the previous term, then add 6. The differences between terms are not constant, and the ratios are not constant either. From the table, the values are getting closer to 12, but they do not follow the simple arithmetic or geometric patterns used in the other questions.

C1 — Savings Account

Amara opens a savings account with £600. Each year, 5% interest is added at the end of the year. Let V_k be the value of the account **after** k **complete** years.

(a) **Solution.** $V_0 = 600$. The sequence starts at $k = 0$ because after 0 complete years no interest has yet been added, so the account still contains the original £600.

(b) The iterative process is

$$V_{k+1} = 1.05V_k, \quad V_0 = 600.$$

(c) The completed table, using exact values and rounding each to the nearest penny, is:

k	0	1	2	3	4
V_k (£)	£600.00	£630.00	£661.50	£694.58	£729.30

Note: $600(1.05)^3 = 694.575$, so $V_3 \approx £694.58$; $600(1.05)^4 = 729.30375$, so $V_4 \approx £729.30$.

(d) The direct formula is

$$V_k = 600(1.05)^k.$$

Therefore

$$V_{10} = 600(1.05)^{10} = 977.33677 \dots \approx £977.34.$$

C2 — Bouncing Ball

A ball is dropped from a height of 4 m. Each bounce reaches 75% of the previous height.

Jess defines H_n as the height reached after n bounces ignoring the initial drop.

Kwame defines H_n as the height after n bounces, where the *original drop height counts as the height after 0 bounces*.

(a) **Jess's process:**

$$H_{n+1} = 0.75H_n, \quad H_1 = 3.$$

The first bounce reaches $4 \times 0.75 = 3$ m, and Jess ignores the initial drop, so she starts at $H_1 = 3$.

(b) **Kwame's process:**

$$H_{n+1} = 0.75H_n, \quad H_0 = 4.$$

Kwame counts the original drop height as $H_0 = 4$ m before any bounces have happened.

(c) **Same:** Both use the same recurrence and the same multiplier: each new height is 75% of the previous height, i.e. multiply by 0.75.

(d) **Different:** The starting index and starting value are different.

$$\text{Jess: } n = 1, H_1 = 3; \quad \text{Kwame: } n = 0, H_0 = 4.$$

(e) **Jess:**

$$H_3 = 3(0.75)^2 = 3 \times 0.5625 = 1.6875 \text{ m.}$$

This represents the height reached after 3 bounces.

Kwame:

$$H_3 = 4(0.75)^3 = 4 \times 0.421875 = 1.6875 \text{ m.}$$

This also represents the height reached after 3 bounces.

(f) **Solution.** Kwame's definition is slightly more natural because it includes the original drop as H_0 , so the formula $H_n = 4(0.75)^n$ works for all $n \geq 0$. Jess's definition is also perfectly valid; both definitions give the same physical bounce heights.

Section D Extension and Challenge Problems

D1 — Working Backwards

The 5th term of an iterative sequence is $U_5 = 48$.

The iterative process is $U_{n+1} = 2U_n$.

(a) **Solution.** Since $U_{n+1} = 2U_n$, working backwards gives

$$U_n = \frac{U_{n+1}}{2}.$$

Therefore:

$$U_4 = \frac{48}{2} = 24, \quad U_3 = \frac{24}{2} = 12, \quad U_2 = \frac{12}{2} = 6, \quad U_1 = \frac{6}{2} = 3.$$

Hence $U_1 = 3$.

(b) The n th term formula is

$$U_n = 3 \cdot 2^{n-1}.$$

Check:

$$U_5 = 3 \cdot 2^4 = 3 \times 16 = 48.$$

D2 — Convergence

Consider the iterative process $F_{j+1} = \frac{F_j}{2} + 6$, $F_1 = 100$

- (a) The completed table, to 2 decimal places, is:

j	1	2	3	4	5	6	7	8
F_j	100.00	56.00	34.00	23.00	17.50	14.75	13.38	12.69

- (b) It appears that F_j is approaching 12.

- (c) **Solution.** At a fixed point,

$$L = \frac{L}{2} + 6.$$

Subtract $\frac{L}{2}$ from both sides:

$$\frac{L}{2} = 6.$$

Therefore

$$L = 12.$$

- (d) **Solution.** The fixed point $L = 12$ is exactly the value that the sequence appears to be approaching. No finite term will be exactly equal to 12, because

$$F_j - 12 = (F_1 - 12) \left(\frac{1}{2}\right)^{j-1} = \frac{88}{2^{j-1}} \neq 0$$

for any finite value of j . The terms get closer and closer to 12 but never actually reach it.