

An Introduction to Indices and Roots

Worked Solutions

Each solution is written out using the method taught in the worked example that comes before the question on the worksheet, and in the same steps.

An **index** tells us how many times a number is multiplied by itself. In 5^3 the number 5 is called the **base** and the small raised number 3 is called the **index**, so 5^3 means $5 \times 5 \times 5$. The plural of index is **indices**. A **root** undoes an index.

Notation: indices and roots

$a^2 = a \times a$	“ a squared”	for example $7^2 = 7 \times 7 = 49$
$a^3 = a \times a \times a$	“ a cubed”	for example $2^3 = 2 \times 2 \times 2 = 8$
$\sqrt{a}$	the square root of a	for example $\sqrt{49} = 7$
$\sqrt[3]{a}$	the cube root of a	for example $\sqrt[3]{8} = 2$

Part 1: Indices and roots of numbers

Q1. Work out each calculation and join it to its answer.

Worked solution. An index is worked out by writing the multiplication out in full. A root is worked out by finding the number that gives the answer when it is multiplied by itself the right number of times.

Number	Calculation	Joins to
1	$5^2 = 5 \times 5 = 25$	D
2	$\sqrt{100} = 10$, because $10 \times 10 = 100$	H
3	$2^3 = 2 \times 2 \times 2 = 8$	B
4	$0.2^2 = 0.2 \times 0.2 = 0.04$	F
5	$\sqrt{0.25} = 0.5$, because $0.5 \times 0.5 = 0.25$	A
6	$10^3 = 10 \times 10 \times 10 = 1000$	E
7	$\sqrt[3]{8} = 2$, because $2 \times 2 \times 2 = 8$	C
8	$1.5^2 = 1.5 \times 1.5 = 2.25$	G

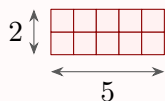
In numbers 4 and 8 the base is a decimal. We multiply the digits first and then count the decimal places. For number 4 we have $2 \times 2 = 4$, and 0.2 has one decimal place and 0.2 has one decimal place, so the answer has two decimal places, which gives 0.04. For number 8 we have $15 \times 15 = 225$, and again there are two decimal places altogether, which gives 2.25.

The joins are 1–D, 2–H, 3–B, 4–F, 5–A, 6–E, 7–C and 8–G.

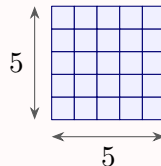
Common mistake: reading the index as a multiplier

The index is **not** a second number to multiply by, so 5^2 is not $5 \times 2 = 10$. In 5^2 the index 2 says how many 5s are multiplied together. In Q2 this mistake gives $6^2 = 12 \text{ m}^2$ and $4^3 = 12 \text{ cm}^3$. Counting the squares in each picture shows the difference.

$$\otimes \quad 5 \times 2 = 10 \text{ squares}$$



$$\checkmark \quad 5^2 = 5 \times 5 = 25 \text{ squares}$$



Q2. Area, side, volume and edge.

Worked solution.

- (a) The patio is a square with sides of length 6 m, so its area is the length of a side multiplied by itself.

$$\text{Area} = 6^2 = 6 \times 6 = 36 \text{ m}^2$$

- (b) The area of the second patio is 144 m^2 , so the length of a side is the number that gives 144 when it is multiplied by itself. That number is the square root of 144.

$$\text{Side} = \sqrt{144} = 12 \text{ m}, \quad \text{because } 12 \times 12 = 144$$

- (c) The box is a cube with edges of length 4 cm, so its volume is the length of an edge multiplied by itself three times.

$$\text{Volume} = 4^3 = 4 \times 4 \times 4 = 64 \text{ cm}^3$$

- (d) The volume of the second box is 125 cm^3 , so the length of an edge is the number that gives 125 when three of them are multiplied together. That number is the cube root of 125.

$$\text{Edge} = \sqrt[3]{125} = 5 \text{ cm}, \quad \text{because } 5 \times 5 \times 5 = 125$$

- (e) The area of the floor tile is 0.25 m^2 , so the length of a side is the number that gives 0.25 when it is multiplied by itself. That number is the square root of 0.25.

$$\text{Side} = \sqrt{0.25} = 0.5 \text{ m}, \quad \text{because } 0.5 \times 0.5 = 0.25$$

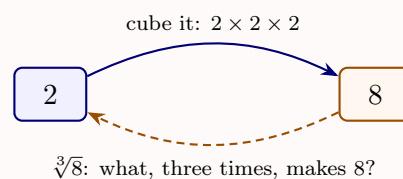
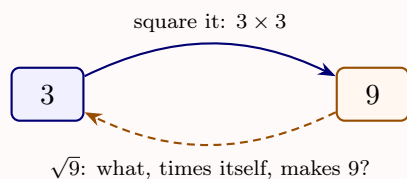
- (f) The square numbers either side of 50 are $7^2 = 49$ and $8^2 = 64$. The area of 50 m^2 is between 49 and 64, so the length of a side is between $\sqrt{49} = 7 \text{ m}$ and $\sqrt{64} = 8 \text{ m}$.

$$7^2 = 49 < 50 < 64 = 8^2, \quad \text{so the side is between 7 m and 8 m.}$$

Because 50 is much closer to 49 than to 64, the side is only a little more than 7 m. On a calculator $\sqrt{50} = 7.07 \dots \text{ m}$, which checks the answer.

Common mistake: dividing by the index to find a root

A root undoes an index, but it does **not** divide by the index. The square root of 144 is not $144 \div 2 = 72$, and the cube root of 125 is not $125 \div 3$. A root asks which number, multiplied by itself the right number of times, rebuilds the number we started with. An answer is checked by multiplying it back: $72 \times 72 = 5184$, which is not 144, but $12 \times 12 = 144$.



Part 2: Indices and algebraic variables

Q3. Write each expression in index form.

Worked solution. We count how many times each algebraic variable appears in the numerator and how many times it appears in the denominator, and we write each count as an index.

(a) The variable a appears 4 times, so $a \times a \times a \times a = a^4$.

(b) We use the three steps of the worked example.

Step 1 — count the numerator. c appears 3 times, so the numerator is c^3 .

Step 2 — count the denominator. d appears 2 times, so the denominator is d^2 .

Step 3 — write the fraction. $\frac{c \times c \times c}{d \times d} = \frac{c^3}{d^2}$

(c) The variable b appears 5 times, so $b \times b \times b \times b \times b = b^5$.

(d) The number 3 multiplies the variable, so it is written in front, and m appears 2 times, which gives $3 \times m \times m = 3m^2$ (remember $3m^2 = 3 \times m \times m$).

(e) The numerator has p appearing 3 times and the denominator has q appearing 4 times, so $\frac{p \times p \times p}{q \times q \times q \times q} = \frac{p^3}{q^4}$.

(f) The numerator is $2 \times x \times x \times x = 2x^3$ and the denominator is $y \times y = y^2$, so $\frac{2 \times x \times x \times x}{y \times y} = \frac{2x^3}{y^2}$.

(g) The numerator has two different variables in it, and each one is counted separately: a appears 2 times and b appears 3 times, so the numerator is a^2b^3 . The denominator is c^2 , which gives $\frac{a \times a \times b \times b \times b}{c \times c} = \frac{a^2b^3}{c^2}$.

(h) The numerator is $t \times t = t^2$ and the denominator is $5 \times u \times u \times u = 5u^3$, so $\frac{t \times t}{5 \times u \times u \times u} = \frac{t^2}{5u^3}$.

Q4. Write each expression out in full as a multiplication.

Worked solution. The index tells us how many times the base is written down in the multiplication.

(a) $x^3 = x \times x \times x$, which is given on the worksheet.

(b) The index is 5 and the base is y , so $y^5 = y \times y \times y \times y \times y$.

(c) The index belongs to the t only, and the 4 multiplies the whole of t^2 , so $4t^2 = 4 \times t \times t$.

(d) The numerator is $m^2 = m \times m$ and the denominator is $n^3 = n \times n \times n$, so $\frac{m^2}{n^3} = \frac{m \times m}{n \times n \times n}$.

(e) The numerator is $a^4 = a \times a \times a \times a$. The denominator is b , which has an index of 1 that is not written, so b appears once and $\frac{a^4}{b} = \frac{a \times a \times a \times a}{b}$.

Common mistake: counting the number in front as part of the index

In Q4(c) the index belongs to the t only, so $4t^2 = 4 \times t \times t$. It is **not** $4t \times 4t$. If $t = 3$ then $4t^2 = 4 \times 3 \times 3 = 36$, but $4t \times 4t = 12 \times 12 = 144$. In the same way, in Q3(d) $3 \times m \times m$ is $3m^2$ and not m^3 , because the index counts only the factors that are m . Each factor below is one tile, and the index counts only the tiles that hold the variable.

the 4 multiplies, it is not a t

$$\begin{array}{c} \downarrow \\ \boxed{4} \underbrace{\boxed{t} \boxed{t}}_{\text{two } t \text{ tiles, so } t^2} = 4t^2 \quad \checkmark \end{array}$$

$$\underbrace{\boxed{4} \boxed{t} \boxed{4} \boxed{t}}_{\text{four tiles, so this is not } 4t^2} = 4t \times 4t = 16t^2 \quad \otimes$$

Part 3: Cancelling factors in a fraction

The **numerator** is the expression on the top of a fraction and the **denominator** is the expression on the bottom. A factor that appears in both is divided out of both, which is called cancelling.

Worked example: simplify $\frac{12}{18}$

Step 1 — write the numerator and the denominator as multiplications. $\frac{12}{18} = \frac{2 \times 2 \times 3}{2 \times 3 \times 3}$

Step 2 — cross out each factor that appears in both. $\frac{12}{18} = \frac{\cancel{2} \times 2 \times \cancel{3}}{\cancel{2} \times \cancel{3} \times 3}$

Step 3 — multiply what is left on the top and on the bottom. $\frac{12}{18} = \frac{2}{3}$

Common mistake: cancelling by column, or cancelling digits

Cancelling pairs **one** factor on the top with **one** factor on the bottom that has the same value, whichever column each of them is in. The pairs are not found by lining the columns up, and one factor cannot cancel two. Digits are not factors either: in $14/21$ the two 1s cannot be crossed out, because $14 = 7 \times 2$ and $21 = 7 \times 3$, so the factor they share is 7.

numerator	$12 =$	$2 \times$	$\boxed{2} \times$	3	each arrow pairs one factor on the top with the one factor on the bottom it is divided out with, whichever column it is in. The dotted factors have no partner.
denominator	$18 =$	$2 \times$	$3 \times$	$\boxed{3}$	

Q5. Simplify each fraction by cancelling.

Worked solution. Each numerator and each denominator is written as a multiplication of its factors first, so that the factors they share can be seen and crossed out.

(a) $8 = 2 \times 2 \times 2$ and $12 = 2 \times 2 \times 3$, and the factors 2 and 2 appear in both.

$$\frac{8}{12} = \frac{2 \times 2 \times 2}{2 \times 2 \times 3} = \frac{\cancel{2} \times \cancel{2} \times 2}{\cancel{2} \times \cancel{2} \times 3} = \frac{2}{3}$$

(b) $15 = 5 \times 3$ and $25 = 5 \times 5$, and the factor 5 appears in both.

$$\frac{15}{25} = \frac{5 \times 3}{5 \times 5} = \frac{\cancel{5} \times 3}{\cancel{5} \times 5} = \frac{3}{5}$$

(c) $18 = 2 \times 3 \times 3$ and $24 = 2 \times 3 \times 2 \times 2$, and the factors 2 and 3 appear in both.

$$\frac{18}{24} = \frac{2 \times 3 \times 3}{2 \times 3 \times 2 \times 2} = \frac{\cancel{2} \times \cancel{3} \times 3}{\cancel{2} \times \cancel{3} \times 2 \times 2} = \frac{3}{4}$$

(d) $14 = 7 \times 2$ and $21 = 7 \times 3$, and the factor 7 appears in both.

$$\frac{14}{21} = \frac{7 \times 2}{7 \times 3} = \frac{\cancel{7} \times 2}{\cancel{7} \times 3} = \frac{2}{3}$$

(e) $45 = 5 \times 3 \times 3$ and $60 = 5 \times 3 \times 2 \times 2$, and the factors 5 and 3 appear in both.

$$\frac{45}{60} = \frac{5 \times 3 \times 3}{5 \times 3 \times 2 \times 2} = \frac{\cancel{5} \times \cancel{3} \times 3}{\cancel{5} \times \cancel{3} \times 2 \times 2} = \frac{3}{4}$$

(f) $24 = 2 \times 2 \times 2 \times 3$ and $36 = 2 \times 2 \times 3 \times 3$, and the factors 2, 2 and 3 appear in both.

$$\frac{24}{36} = \frac{2 \times 2 \times 2 \times 3}{2 \times 2 \times 3 \times 3} = \frac{\cancel{2} \times \cancel{2} \times 2 \times \cancel{3}}{\cancel{2} \times \cancel{2} \times \cancel{3} \times 3} = \frac{2}{3}$$

(g) 12 of the 28 pupils walk, so the fraction of the class who walk is $\frac{12}{28}$. Here $12 = 2 \times 2 \times 3$ and $28 = 2 \times 2 \times 7$, and the factors 2 and 2 appear in both.

$$\frac{12}{28} = \frac{2 \times 2 \times 3}{2 \times 2 \times 7} = \frac{\cancel{2} \times \cancel{2} \times 3}{\cancel{2} \times \cancel{2} \times 7} = \frac{3}{7}$$

(h) 45 g of the 120 g is sugar, so the fraction that is sugar is $\frac{45}{120}$. Here $45 = 3 \times 3 \times 5$ and $120 = 2 \times 2 \times 2 \times 3 \times 5$, and the factors 3 and 5 appear in both.

$$\frac{45}{120} = \frac{3 \times 3 \times 5}{2 \times 2 \times 2 \times 3 \times 5} = \frac{3 \times \cancel{3} \times \cancel{5}}{2 \times 2 \times 2 \times \cancel{3} \times \cancel{5}} = \frac{3}{2 \times 2 \times 2} = \frac{3}{8}$$

Q6. Simplify each fraction by cancelling, then write the answer in index form.

Worked solution. The numbers and the variables are cancelled separately. Where every factor on the bottom is crossed out, the denominator is written as 1 before the fraction is written as its numerator.

- (a) Two of the four a s on the top cancel with the two a s on the bottom, and two a s are left on the top.

$$\frac{a \times a \times a \times a}{a \times a} = \frac{\cancel{a} \times \cancel{a} \times a \times a}{\cancel{a} \times \cancel{a}} = \frac{a \times a}{1} = a^2$$

- (b) Two of the five b s on the top cancel with the two b s on the bottom, and three b s are left on the top.

$$\frac{b \times b \times b \times b \times b}{b \times b} = \frac{\cancel{b} \times \cancel{b} \times b \times b \times b}{\cancel{b} \times \cancel{b}} = \frac{b \times b \times b}{1} = b^3$$

- (c) One factor of 2 and one factor of m cancel.

$$\frac{2 \times 2 \times m \times m}{2 \times m} = \frac{\cancel{2} \times 2 \times \cancel{m} \times m}{\cancel{2} \times \cancel{m}} = \frac{2 \times m}{1} = 2m$$

- (d) The factor 5 appears on the top and on the bottom, so it cancels, but the factor 3 appears on the top only, so it stays. One factor of p cancels as well.

$$\frac{3 \times 5 \times p \times p \times p}{5 \times p} = \frac{3 \times \cancel{5} \times \cancel{p} \times p \times p}{\cancel{5} \times \cancel{p}} = \frac{3 \times p \times p}{1} = 3p^2$$

- (e) One factor of x and one factor of y cancel. Here a factor of y is left on the bottom, so the denominator is y and not 1.

$$\frac{x \times x \times y}{x \times y \times y} = \frac{\cancel{x} \times x \times \cancel{y}}{\cancel{x} \times y \times y} = \frac{x}{y}$$

- (f) One factor of 3 and one factor of t cancel, and a factor of 3 is left on the bottom.

$$\frac{2 \times 3 \times t \times t}{3 \times 3 \times t} = \frac{2 \times \cancel{3} \times \cancel{t} \times t}{\cancel{3} \times 3 \times \cancel{t}} = \frac{2 \times t}{3} = \frac{2t}{3}$$

- (g) Here $4 = 2 \times 2$ and $6 = 2 \times 3$, so one factor of 2 cancels. One factor of x and one factor of y cancel as well.

$$\frac{4 \times x \times x \times y}{6 \times x \times y \times y} = \frac{\cancel{2} \times 2 \times \cancel{x} \times x \times \cancel{y}}{\cancel{2} \times 3 \times \cancel{x} \times y \times y} = \frac{2 \times x}{3 \times y} = \frac{2x}{3y}$$

- (h) One factor of 5, one factor of a and one factor of b cancel. A factor of a is left on the bottom, so the denominator is a and not 1.

$$\frac{5 \times 5 \times a \times b \times b}{5 \times a \times a \times b} = \frac{\cancel{5} \times 5 \times \cancel{a} \times \cancel{b} \times b}{\cancel{5} \times \cancel{a} \times a \times \cancel{b}} = \frac{5 \times b}{a} = \frac{5b}{a}$$

- (i) **Model answer.** Two factors of a cancel from the top and from the bottom. Every factor on the top has cancelled, so the numerator is 1, and one factor of a is left on the bottom. Lena has put the a that is left on the top instead of on the bottom. The correct answer is $\frac{1}{a}$.

$$\frac{a \times a}{a \times a \times a} = \frac{\cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a} \times a} = \frac{1}{a}$$

Also accepted. An answer that shows the mistake by substituting a number is also correct. For example $a = 2$ gives $\frac{2 \times 2}{2 \times 2 \times 2} = \frac{4}{8} = \frac{1}{2}$, which is $\frac{1}{a}$ and not a .

Part 4: Simplifying fractions written with indices

Each power is written out in full as a multiplication, the factors that appear on the top and on the bottom are cancelled, and what is left is written in index form.

Q7. Simplify each fraction and join it to its answer.

Worked solution.

- Two of the five a s cancel, leaving three. $\frac{a^5}{a^2} = \frac{\cancel{a} \times \cancel{a} \times a \times a \times a}{\cancel{a} \times \cancel{a}} = \frac{a \times a \times a}{1} = a^3$, which joins to **D**.
- Every factor on the top and on the bottom cancels, so the numerator is 1 and the denominator is 1. $\frac{a^2}{a^2} = \frac{\cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a}} = \frac{1}{1} = 1$, which joins to **F**.
- Here $6 = 2 \times 3$, so a factor of 2 cancels as well as two of the a s. $\frac{6a^4}{2a^2} = \frac{\cancel{2} \times 3 \times \cancel{a} \times \cancel{a} \times a \times a}{\cancel{2} \times \cancel{a} \times \cancel{a}} = \frac{3 \times a \times a}{1} = 3a^2$, which joins to **H**.
- One factor of a and the single factor of b cancel. $\frac{a^3b}{ab} = \frac{\cancel{a} \times a \times a \times \cancel{b}}{\cancel{a} \times \cancel{b}} = \frac{a \times a}{1} = a^2$, which joins to **G**.
- Here $4 = 2 \times 2$ and $8 = 2 \times 2 \times 2$, so two factors of 2 cancel and one factor of 2 is left on the bottom. $\frac{4a^2}{8a} = \frac{\cancel{2} \times \cancel{2} \times a \times a}{\cancel{2} \times \cancel{2} \times 2 \times a} = \frac{a}{2}$, which joins to **E**.
- Both factors of a cancel, and one of the three b s cancels. $\frac{a^2b^3}{a^2b} = \frac{\cancel{a} \times \cancel{a} \times \cancel{b} \times b \times b}{\cancel{a} \times \cancel{a} \times \cancel{b}} = \frac{b \times b}{1} = b^2$, which joins to **A**.
- Here $10 = 5 \times 2$, so the factor 5 cancels and all three a s cancel. $\frac{10a^3}{5a^3} = \frac{\cancel{5} \times 2 \times \cancel{a} \times \cancel{a} \times \cancel{a}}{\cancel{5} \times \cancel{a} \times \cancel{a} \times \cancel{a}} = \frac{2}{1} = 2$, which joins to **C**.
- Two of the six a s cancel, leaving four. $\frac{a^6}{a^2} = \frac{\cancel{a} \times \cancel{a} \times a \times a \times a \times a}{\cancel{a} \times \cancel{a}} = \frac{a \times a \times a \times a}{1} = a^4$, which joins to **B**.

The joins are 1–D, 2–F, 3–H, 4–G, 5–E, 6–A, 7–C and 8–B.

Q8. Writing fractions that simplify to a given expression.

- (a) **Model answer.** Three fractions that simplify to $2a^2$ are $\frac{2a^3}{a}$, $\frac{4a^3}{2a}$ and $\frac{6a^5}{3a^3}$. The second and the third both have a number other than 1 in the denominator. Each one is checked by cancelling.

$$\begin{aligned}\frac{2a^3}{a} &= \frac{2 \times \cancel{a} \times a \times a}{\cancel{a}} = \frac{2 \times a \times a}{1} = 2a^2 \\ \frac{4a^3}{2a} &= \frac{\cancel{2} \times 2 \times \cancel{a} \times a \times a}{\cancel{2} \times \cancel{a}} = \frac{2 \times a \times a}{1} = 2a^2 \\ \frac{6a^5}{3a^3} &= \frac{\cancel{3} \times 2 \times \cancel{a} \times \cancel{a} \times \cancel{a} \times a \times a}{\cancel{3} \times \cancel{a} \times \cancel{a} \times \cancel{a}} = \frac{2 \times a \times a}{1} = 2a^2\end{aligned}$$

Also accepted. Any fraction whose numerator is its denominator multiplied by $2a^2$ is correct, for example $\frac{10a^4}{5a^2}$, $\frac{2a^4}{a^2}$ or $\frac{8a^3}{4a}$. One of the three fractions must have a number other than 1 in its denominator.

- (b) **Model answer.** One fraction that simplifies to $\frac{a^2}{b}$ is $\frac{a^3}{ab}$, because one factor of a cancels from the top and from the bottom.

$$\frac{a^3}{ab} = \frac{\cancel{a} \times a \times a}{\cancel{a} \times b} = \frac{a^2}{b}$$

Also accepted. Any fraction whose numerator has exactly two more factors of a than its denominator, and whose denominator has exactly one more factor of b than its numerator, is correct, as long as it is not $\frac{a^2}{b}$ itself. Examples are $\frac{a^2b}{b^2}$, $\frac{a^4}{a^2b}$ and $\frac{a^4b}{a^2b^2}$.

- (c) **Model answer.** One fraction that uses x , y and z and simplifies to $\frac{1}{xy^2}$ is $\frac{z}{xy^2z}$. The factor z cancels from the top and from the bottom, which leaves 1 on the top.

$$\frac{z}{xy^2z} = \frac{\cancel{z}}{x \times y \times y \times \cancel{z}} = \frac{1}{xy^2}$$

Also accepted. Any fraction in which every factor on the top also appears on the bottom, and the factors left on the bottom are one x and two y s, is correct. Examples are $\frac{xz}{x^2y^2z}$ and $\frac{yz^2}{xy^3z^2}$.

- (d) **Model answer.** Ben has crossed out every factor correctly, but he has then written that nothing is left. Cancelling divides the numerator and the denominator by the same factor, and a factor divided by itself is 1, not 0. When every factor on the top cancels the numerator is 1, and when every factor on the bottom cancels the denominator is 1, so the correct answer is $\frac{1}{1} = 1$.

$$\frac{3a^2}{3a^2} = \frac{\cancel{3} \times \cancel{a} \times \cancel{a}}{\cancel{3} \times \cancel{a} \times \cancel{a}} = \frac{1}{1} = 1$$

Also accepted. Any answer that says a number or a variable divided by itself is 1, and that gives the correct answer of 1, is correct. An answer that checks the result by substituting a number, for example $a = 2$ giving $\frac{3 \times 2 \times 2}{3 \times 2 \times 2} = \frac{12}{12} = 1$, is also correct.

Common mistake: a crossed-out factor leaves nothing

Crossing a factor out does not rub it off the page. A factor divided by itself is 1, so a crossed-out factor leaves a 1 behind. It never leaves nothing, and it never leaves a 0.

$$\frac{12}{18} = \frac{\overset{1}{\cancel{2}} \times 2 \times \overset{1}{\cancel{3}}}{\overset{1}{\cancel{2}} \times \overset{1}{\cancel{3}} \times 3} = \frac{1 \times 2 \times 1}{1 \times 1 \times 3} = \frac{2}{3} \qquad \frac{a^2}{a^2} = \frac{\overset{1}{\cancel{a}} \times \overset{1}{\cancel{a}}}{\overset{1}{\cancel{a}} \times \overset{1}{\cancel{a}}} = \frac{1 \times 1}{1 \times 1} = \frac{1}{1} = 1$$

Q9. A quicker way to simplify a fraction such as $\frac{a^7}{a^3}$.

- (a) **Worked solution.** Each power is written out in full, and the factors that appear on the top and on the bottom are cancelled.

$$\frac{a^7}{a^3} = \frac{\cancel{a} \times \cancel{a} \times \cancel{a} \times a \times a \times a \times a}{\cancel{a} \times \cancel{a} \times \cancel{a}} = \frac{a \times a \times a \times a}{1} = a^4$$

In the same way, four of the nine b s cancel, leaving five, so $\frac{b^9}{b^4} = b^5$. Five of the six c s cancel, leaving one, so $\frac{c^6}{c^5} = c^1 = c$.

- (b) **Model answer.** Subtract the index on the bottom from the index on the top, and keep the same base. For example $7 - 3 = 4$, so $\frac{a^7}{a^3} = a^4$. This works because each factor on the bottom cancels one factor on the top, so the number of factors left on the top is the index on the top take away the index on the bottom.

Also accepted. Any answer that describes taking the index on the bottom away from the index on the top is correct.

- (c) **Worked solution.**

$$\frac{x^{20}}{x^{13}} = x^{20-13} = x^7 \qquad \frac{y^{100}}{y^{99}} = y^{100-99} = y^1 = y$$

- (d) **Worked solution.** The quicker way gives $\frac{a^3}{a^3} = a^{3-3} = a^0$. Cancelling gives

$$\frac{a^3}{a^3} = \frac{\cancel{a} \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a} \times \cancel{a}} = \frac{1}{1} = 1$$

Both ways simplify the same fraction, so they must give the same answer. This suggests that $a^0 = 1$.

Part 5: Extension: factorials, negative numbers and the fourth dimension

The **factorial** $n!$ means multiply together every whole number from n down to 1.

Q10. Work out each factorial.

Worked solution.

(a) $3! = 3 \times 2 \times 1 = 6$

(b) $4! = 4 \times 3 \times 2 \times 1 = 24$

(c) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

(d) $1! = 1$, because the only whole number from 1 down to 1 is 1 itself.

Worked example: simplify $\frac{6!}{4!}$

Both factorials are written out in full. Every whole number from 4 downwards appears on the top and on the bottom, so all of those factors cancel.

$$\frac{6!}{4!} = \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{6 \times 5}{1} = 6 \times 5 = 30$$

Q11. Simplify each expression by writing the factorials out in full and cancelling.

Worked solution. The factorials are not worked out first. Every factor that appears on the top and on the bottom is crossed out, and what is left is multiplied.

(a) Every whole number from 3 downwards appears on the top and on the bottom, so all of those factors cancel and the denominator is 1.

$$\frac{5!}{3!} = \frac{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{5 \times 4}{1} = 5 \times 4 = 20$$

(b) Every whole number from 9 downwards cancels, and only the 10 is left on the top.

$$\frac{10!}{9!} = \frac{10 \times \cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{9} \times \cancel{8} \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{10}{1} = 10$$

(c) Every whole number from 6 downwards cancels, leaving the 8 and the 7 on the top.

$$\frac{8!}{6!} = \frac{8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = \frac{8 \times 7}{1} = 8 \times 7 = 56$$

(d) The denominator is $5! \times 2!$. The $5!$ cancels with the whole numbers from 5 downwards on the top, but the $2!$ is a separate factorial and stays on the bottom.

$$\frac{7!}{5! \times 2!} = \frac{7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{(\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}) \times (2 \times 1)} = \frac{7 \times 6}{2 \times 1} = \frac{42}{2} = 21$$

(e) Every whole number from 98 downwards appears on the top and on the bottom, so all of those factors cancel, and only the 100 and the 99 are left on the top. There are too many factors to write out, so the three dots stand for the whole numbers in between.

$$\frac{100!}{98!} = \frac{100 \times 99 \times \cancel{98} \times \cancel{97} \times \cdots \times \cancel{1}}{\cancel{98} \times \cancel{97} \times \cdots \times \cancel{1}} = \frac{100 \times 99}{1} = 9900$$

- (f) The $10!$ cancels with the whole numbers from 10 downwards on the top, and the $2!$ stays on the bottom, as in part (d).

$$\frac{12!}{10! \times 2!} = \frac{12 \times 11 \times \cancel{10} \times \cdots \times \cancel{1}}{(\cancel{10} \times \cdots \times \cancel{1}) \times (2 \times 1)} = \frac{12 \times 11}{2 \times 1} = \frac{132}{2} = 66$$

Q12. Explain why $\frac{n!}{(n-1)!}$ always simplifies to n .

Model answer. We write both factorials out in full. The numerator is $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$ and the denominator is $(n-1)! = (n-1) \times (n-2) \times \cdots \times 1$. Every factor of the denominator also appears in the numerator, so every one of those factors cancels. The only factor left on the top is n , and every factor on the bottom has cancelled, so the denominator is 1. A fraction with a denominator of 1 is equal to its numerator, so the answer is n .

$$\frac{n!}{(n-1)!} = \frac{n \times \cancel{(n-1)} \times \cancel{(n-2)} \times \cdots \times \cancel{1}}{\cancel{(n-1)} \times \cancel{(n-2)} \times \cdots \times \cancel{1}} = \frac{n}{1} = n$$

Also accepted. An answer that uses $n! = n \times (n-1)!$ and then cancels the factor $(n-1)!$ is correct, because it gives $\frac{n!}{(n-1)!} = \frac{n \times \cancel{(n-1)!}}{\cancel{(n-1)!}} = \frac{n}{1} = n$. An answer that checks the result with numbers, for example $\frac{5!}{4!} = 5$ and $\frac{10!}{9!} = 10$, supports the explanation but is not an explanation on its own.

Q13. Powers of negative numbers.

- (a) **Worked solution.** The brackets show that the whole negative number is multiplied by itself, so every factor in the multiplication is negative.

Expression	Written out in full	Answer
$(-2)^2$	$(-2) \times (-2)$	4
$(-2)^3$	$(-2) \times (-2) \times (-2)$	-8
$(-2)^4$	$(-2) \times (-2) \times (-2) \times (-2)$	16
$(-3)^2$	$(-3) \times (-3)$	9
$(-3)^3$	$(-3) \times (-3) \times (-3)$	-27

Each multiplication is worked out two factors at a time. For $(-2)^4$ we have $(-2) \times (-2) = 4$, then $4 \times (-2) = -8$, then $-8 \times (-2) = 16$. For $(-3)^3$ we have $(-3) \times (-3) = 9$, then $9 \times (-3) = -27$.

- (b) **Model answer.** When the index is even the answer is positive, and when the index is odd the answer is negative.

Also accepted. An answer that states the same pattern using the entries in the table, for example that $(-2)^2$, $(-2)^4$ and $(-3)^2$ are positive while $(-2)^3$ and $(-3)^3$ are negative, is correct.

- (c) **Model answer.** A negative number multiplied by a negative number gives a positive answer, so the negative signs pair up. When the index is even every negative sign has a partner, no negative sign is left over, and the answer is positive. When the index is odd one negative sign is left over after the pairs have been made, and that sign makes the answer negative.

Also accepted. An answer that shows the pairing on an example, for example $(-2)^4 = ((-2) \times (-2)) \times ((-2) \times (-2)) = 4 \times 4 = 16$ and $(-2)^3 = ((-2) \times (-2)) \times (-2) = 4 \times (-2) = -8$, is correct.

Common mistake: deciding the sign without counting

Writing $(-2)^4 = -16$ or $(-3)^2 = -9$ is a common error. The sign of a power of a negative number is found by pairing the minus signs, not by guessing. A pair of minus signs multiplies to give a plus, so the sign depends only on whether a minus sign is left over. The brackets matter: $(-3)^2 = (-3) \times (-3) = 9$, but $-3^2 = -(3 \times 3) = -9$, because without brackets the index belongs to the 3 only.

$$\begin{array}{ll}
 (-2)^2 & \begin{array}{c} \text{Ⓜ} \quad \text{Ⓜ} \end{array} \quad \text{one pair, nothing left over} \rightarrow \text{positive: 4} \\
 (-2)^3 & \begin{array}{c} \text{Ⓜ} \quad \text{Ⓜ} \quad \text{Ⓜ} \end{array} \quad \text{one left over} \rightarrow \text{negative: -8} \\
 (-2)^4 & \begin{array}{c} \text{Ⓜ} \quad \text{Ⓜ} \quad \text{Ⓜ} \quad \text{Ⓜ} \end{array} \quad \text{two pairs, nothing left over} \rightarrow \text{positive: 16}
 \end{array}$$

Q14. Roots of negative numbers.

(a) **Worked solution.**

$$3 \times 3 = 9 \quad \text{and} \quad (-3) \times (-3) = 9$$

The two numbers that give 9 when they are multiplied by themselves are 3 and -3 . The symbol $\sqrt{}$ means the positive square root only, so $\sqrt{9} = 3$.

(b) **Model answer.** The calculator shows an error message, such as “Math ERROR”. No number multiplied by itself gives a negative answer, because a positive number multiplied by a positive number is positive, and a negative number multiplied by a negative number is also positive. There is therefore no number whose square is -9 , and the calculator cannot give an answer.

Also accepted. An answer that names the exact message the calculator shows, or that says the calculator gives no answer, is correct as long as it gives the reason that a number multiplied by itself is never negative.

(c) **Worked solution.**

$$(-2) \times (-2) \times (-2) = 4 \times (-2) = -8$$

The cube root of -8 is the number that gives -8 when three of them are multiplied together, so $\sqrt[3]{-8} = -2$.

(d) **Model answer.** Sam is not right. A negative number has no square root, because the index 2 is even and an even index always gives a positive answer, which is what part (b) shows. A negative number does have a cube root, because the index 3 is odd and an odd index can give a negative answer, which is what part (c) shows. A root of a negative number can therefore be found when the root is an odd one, such as the cube root, but not when the root is an even one, such as the square root.

Also accepted. An answer that says Sam is right about square roots only, and supports this with the two examples $\sqrt{-9}$ having no answer and $\sqrt[3]{-8} = -2$, is correct.

Q15. The tesseract.

- (a) **Worked solution.** Each new dimension is made by sliding the shape, and every corner of the old shape is copied to a new position, so the number of corners doubles each time. On the picture of the cube there are 4 corners on the front square and 4 on the back square, which is 8 corners. On the picture of the tesseract there are 8 corners on the large cube and 8 corners on the small cube, which is 16 corners.

Shape	point	line	square	cube	tesseract
Dimensions	0	1	2	3	4
Corners	1	2	4	8	16

- (b) **Worked solution.** Each number of corners is written as a power of 2, because the number of corners doubles with each new dimension.

$$\text{Square: } 4 = 2 \times 2 = 2^2 \quad \text{Cube: } 8 = 2 \times 2 \times 2 = 2^3 \quad \text{Tesseract: } 16 = 2 \times 2 \times 2 \times 2 = 2^4$$

The index is the number of dimensions of the shape.

- (c) **Worked solution.** The number of corners doubles with each new dimension, and the index is the number of dimensions, so the five-dimensional shape has

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32 \text{ corners.}$$

- (d) **Worked solution.** We keep doubling from 2 until we reach 1024, and count how many 2s have been multiplied together.

$$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024$$

There are ten numbers in the list, so $1024 = 2^{10}$, and the shape with 1024 corners has 10 dimensions.

- (e) **Worked solution.**

$$3^4 = 3 \times 3 \times 3 \times 3 = 81 \text{ cm}^4$$

- (f) **Worked solution.** The fourth root of 16 is the number that gives 16 when four of them are multiplied together.

$$2 \times 2 \times 2 \times 2 = 16, \quad \text{so} \quad \sqrt[4]{16} = 2 \text{ cm}$$

- (g) **Model answer.** Working the multiplication out two factors at a time gives $(-2) \times (-2) = 4$, then $4 \times (-2) = -8$, then $-8 \times (-2) = 16$, so $(-2)^4 = 16$ as well. The index 4 is even, so the negative signs pair up and none is left over. The edge of the tesseract must still be 2 cm, because the edge is a length, and a length cannot be negative.

Also accepted. An answer that shows the pairing as $(-2)^4 = ((-2) \times (-2)) \times ((-2) \times (-2)) = 4 \times 4 = 16$ is correct, and so is an answer that says $\sqrt[4]{}$ means the positive fourth root, in the same way that $\sqrt{}$ means the positive square root.

- (h) **Worked solution.** The fourth root of 81 is the number that gives 81 when four of them are multiplied together.

$$3 \times 3 \times 3 \times 3 = 81, \quad \text{so} \quad \sqrt[4]{81} = 3 \text{ cm}$$

This is the calculation in part (e) done the other way round.

- (i) **Model answer.** The square has 4 edges and $4 = 2 \times 2^1$, and the tesseract has 32 edges and $32 = 4 \times 2^3$. Written this way the three shapes give

$$\text{Square: } 4 = 2 \times 2^1 \quad \text{Cube: } 12 = 3 \times 2^2 \quad \text{Tesseract: } 32 = 4 \times 2^3$$

The number in front is the number of dimensions, and the index is one less than the number of dimensions. The five-dimensional shape therefore has $5 \times 2^4 = 5 \times 16 = 80$ edges.

Also accepted. An answer that states the rule for a shape of n dimensions as $n \times 2^{n-1}$ is correct. An answer that reaches 80 by noticing that the number in front goes up by 1 and the power of 2 doubles each time, so that the edges go 4, 12, 32, 80, is also correct.

- (j) **Worked solution.** The number of corners is 2 to the power of the number of dimensions, so the six-dimensional shape has

$$2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64 \text{ corners.}$$

The number of edges is the number of dimensions multiplied by 2 to the power of one less than the number of dimensions, so it has

$$6 \times 2^5 = 6 \times 32 = 192 \text{ edges.}$$