

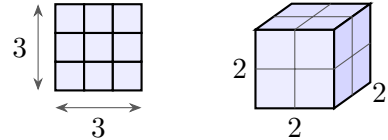
Indices and Roots

An **index** tells us how many times a number is multiplied by itself. So 5^3 means $5 \times 5 \times 5$. The plural of index is **indices**. A **root** undoes an index.

Part 1: Indices and roots of numbers

3^2 is read “three squared”. 2^3 is read “two cubed”.

The **square root** of a number is the value that gives that number when it (the square root) is multiplied by itself, so $\sqrt{9} = 3$. The **cube root** of a number is the value that gives that number when three of them are multiplied together, so $\sqrt[3]{8} = 2$.



$$3^2 = 3 \times 3 = 9 \quad 2^3 = 2 \times 2 \times 2 = 8$$

Notation: indices and roots

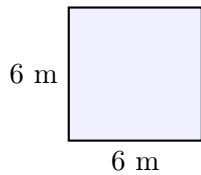
$a^2 = a \times a$	“a squared”	for example $7^2 = 7 \times 7 = 49$
$a^3 = a \times a \times a$	“a cubed”	for example $2^3 = 2 \times 2 \times 2 = 8$
$\sqrt{a}$	the square root of a	for example $\sqrt{49} = 7$
$\sqrt[3]{a}$	the cube root of a	for example $\sqrt[3]{8} = 2$
An index multiplies the base by itself, and a root finds the base again.		

Q1. Work out each calculation on the left and draw a line joining it to its answer on the right. Check your answers on a calculator afterwards.

1.	5^2	0.5	A
2.	$\sqrt{100}$	8	B
3.	2^3	2	C
4.	0.2^2	25	D
5.	$\sqrt{0.25}$	1000	E
6.	10^3	0.04	F
7.	$\sqrt[3]{8}$	2.25	G
8.	1.5^2	10	H

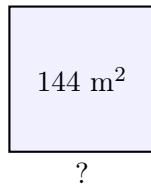
Q2. The shapes are not drawn to scale.

- (a) A square patio has sides of length 6 m. Work out its area.



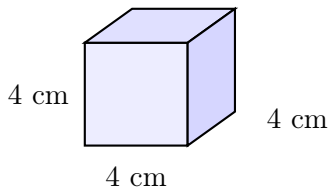
Area = $6^2 = 6 \times 6 =$ _____ m^2

- (b) A second square patio has an area of 144 m^2 . Work out the length of one of its sides.



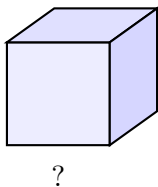
Side = $\sqrt{144}$ = _____ m

- (c) A storage box is a cube with edges of length 4 cm. Work out its volume.



Volume = $4^3 =$ _____ cm^3

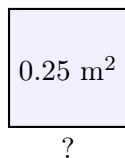
- (d) A second storage box is a cube with a volume of 125 cm^3 . Work out the length of one of its edges.



Edge = _____ cm

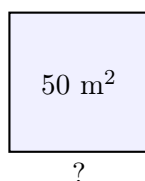
Volume = 125 cm³

- (e) A square floor tile has an area of 0.25 m^2 . Work out the length of one of its sides.



Side = $\sqrt{0.25} =$ _____ m

- (f) A square garden has an area of 50 m^2 . Without a calculator, find the two whole numbers that the length of one side lies between. Explain how you know.



The side is between _____ m and _____ m, because

Part 2: Indices and algebraic variables

An **algebraic variable**, such as a , x or m , could be any number. Just like $5^3 = 5 \times 5 \times 5$, a^3 means $a \times a \times a$.

In $5a^2$ the index belongs to the a only, so $5a^2 = 5 \times a \times a$. It is **not** $5a \times 5a$. If $a = 2$ then $5a^2 = 5 \times 2 \times 2 = 20$, but $5a \times 5a = 10 \times 10 = 100$.

Worked example: writing a fraction in index form

Write $\frac{a \times a \times a}{b \times b}$ in index form.

Step 1 — count the numerator. a appears 3 times, so the numerator is a^3 .

Step 2 — count the denominator. b appears 2 times, so the denominator is b^2 .

Step 3 — write the fraction. $\frac{a \times a \times a}{b \times b} = \frac{a^3}{b^2}$

Q3. Write each expression in index form. Fill in the gaps in parts (a) and (b), then simplify the remaining expressions in the same way.

(a) $a \times a \times a \times a = a^{\square}$

(b) $\frac{c \times c \times c}{d \times d} = \frac{c^{\square}}{d^{\square}}$

(c) $b \times b \times b \times b \times b$

(e) $\frac{p \times p \times p}{q \times q \times q \times q}$

(g) $\frac{a \times a \times b \times b \times b}{c \times c}$

(d) $3 \times m \times m$

(f) $\frac{2 \times x \times x \times x}{y \times y}$

(h) $\frac{t \times t}{5 \times u \times u \times u}$

Q4. Write each expression out in full as a multiplication. The first one is done for you.

(a) $x^3 = x \times x \times x$

(b) $y^5 = \underline{\hspace{2cm}}$

(c) $4t^2 = \underline{\hspace{2cm}}$

(d) $\frac{m^2}{n^3} = \underline{\hspace{2cm}}$

(e) $\frac{a^4}{b} = \underline{\hspace{2cm}}$

Part 3: Cancelling factors in a fraction

A fraction stays the same value when the numerator and the denominator are both divided by the same number. Since the new fraction is **equivalent** to the original.

When the numerator and the denominator are written as multiplications, a factor that appears in both can be divided out of both. This is called **cancelling**, and it is written by crossing the factor out on the top and on the bottom.

Worked example: simplify $\frac{12}{18}$

Step 1 — write the numerator and the denominator as multiplications. $\frac{12}{18} = \frac{2 \times 2 \times 3}{2 \times 3 \times 3}$

Step 2 — cross out each factor that appears in both. $\frac{12}{18} = \frac{\cancel{2} \times 2 \times \cancel{3}}{\cancel{2} \times \cancel{3} \times 3}$

Step 3 — multiply what is left on the top and on the bottom. $\frac{12}{18} = \frac{2}{3}$

Q5. Simplify each fraction by cancelling. Fill in the gaps in parts (a) and (b), then do parts (c) to (f) in the same way.

(a) $\frac{8}{12} = \frac{2 \times 2 \times 2}{2 \times 2 \times 3} = \frac{\cancel{2} \times \cancel{2} \times 2}{\cancel{2} \times \cancel{2} \times 3} = \frac{\boxed{}}{\boxed{}}$

(b) $\frac{15}{25} = \frac{5 \times 3}{5 \times \boxed{}} = \frac{\boxed{}}{\boxed{}}$

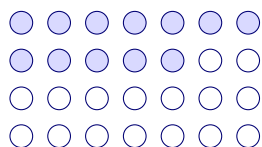
(c) $\frac{18}{24}$

(d) $\frac{14}{21}$

(e) $\frac{45}{60}$

(f) $\frac{24}{36}$

(g) 12 of the 28 pupils in a class walk to school. The picture shows the class, with the pupils who walk shaded. Write the fraction of the class who walk to school in its simplest form.



Fraction who walk = $\frac{12}{28} = \underline{\hspace{2cm}}$

(h) A cake mixture weighs 120 g, and 45 g of it is sugar. Write the fraction of the mixture that is sugar in its simplest form.

Algebraic variables cancel in the same way as numbers. When every factor on the bottom cancels, the denominator is 1, not nothing, a fraction with a denominator of 1 is equal to its numerator.

Worked example: cancelling with numbers and variables

$$\frac{a \times a \times a}{a \times a} = \frac{\cancel{a} \times \cancel{a} \times a}{\cancel{a} \times \cancel{a}} = \frac{a}{1} = a \quad \text{every factor on the bottom cancels, leaving 1}$$

$$\frac{2 \times 3 \times a \times a}{2 \times a} = \frac{\cancel{2} \times 3 \times \cancel{a} \times a}{\cancel{2} \times \cancel{a}} = \frac{3 \times a}{1} = 3a \quad \text{the numbers and the variables are cancelled separately}$$

Q6. Simplify each fraction by cancelling, then write the answer in index form. Fill in the gaps in part (a) first.

(a) $\frac{a \times a \times a \times a}{a \times a} = \frac{\cancel{a} \times \cancel{a} \times a \times a}{\cancel{a} \times \cancel{a}} = \frac{a \times a}{1} = a^{\square}$

(b) $\frac{b \times b \times b \times b \times b}{b \times b} =$

(c) $\frac{2 \times 2 \times m \times m}{2 \times m} =$

(d) $\frac{3 \times 5 \times p \times p \times p}{5 \times p} =$

(e) $\frac{x \times x \times y}{x \times y \times y} =$

(f) $\frac{2 \times 3 \times t \times t}{3 \times 3 \times t} =$

(g) $\frac{4 \times x \times x \times y}{6 \times x \times y \times y} =$

(h) $\frac{5 \times 5 \times a \times b \times b}{5 \times a \times a \times b} =$

(i) Lena simplifies $\frac{a \times a}{a \times a \times a}$ and writes a . Explain the mistake Lena has made, and write down the correct answer.

Part 4: Simplifying fractions written with indices

A fraction written with indices is simplified in the same way. We write each power out in full as a multiplication, cancel the factors that appear on the top and on the bottom, and write what is left in index form.

Worked example: simplifying fractions written with indices

$$\frac{a^5}{a^2} = \frac{\cancel{a} \times \cancel{a} \times a \times a \times a}{\cancel{a} \times \cancel{a}} = a^3 \quad \text{two of the five } a\text{s cancel, leaving three}$$

$$\frac{6a^3}{2a} = \frac{\cancel{2} \times 3 \times \cancel{a} \times a \times a}{\cancel{2} \times \cancel{a}} = 3a^2 \quad 6 = 2 \times 3, \text{ so a factor of 2 cancels as well}$$

Q7. Simplify each fraction on the left. Draw a line joining it to its answer on the right.

- | | | | |
|----|-----------------------|---------------|---|
| 1. | $\frac{a^5}{a^2}$ | b^2 | A |
| 2. | $\frac{a^2}{a^2}$ | a^4 | B |
| 3. | $\frac{6a^4}{2a^2}$ | 2 | C |
| 4. | $\frac{a^3b}{ab}$ | a^3 | D |
| 5. | $\frac{4a^2}{8a}$ | $\frac{a}{2}$ | E |
| 6. | $\frac{a^2b^3}{a^2b}$ | 1 | F |
| 7. | $\frac{10a^3}{5a^3}$ | a^2 | G |
| 8. | $\frac{a^6}{a^2}$ | $3a^2$ | H |

- Q8.** (a) Write down three different fractions that simplify to $2a^2$. One of them must have a number other than 1 in the denominator.

- (b) Write down a fraction (not equal to $\frac{a^2}{b}$) using the variables a and b that simplifies to $\frac{a^2}{b}$.

- (c) Write down a fraction using the variables x , y and z that simplifies to $\frac{1}{xy^2}$.

- (d) Ben says that $\frac{3a^2}{3a^2} = 0$, because every factor on the top and on the bottom cancels and nothing is left. Explain the mistake Ben has made, and write down the correct answer.

- Q9.** In this question we find a quicker way to simplify a fraction such as $\frac{a^7}{a^3}$.

- (a) Simplify each fraction by writing it out in full and cancelling. Give each answer in index form.

$$\frac{a^7}{a^3} = \underline{\hspace{2cm}}$$

$$\frac{b^9}{b^4} = \underline{\hspace{2cm}}$$

$$\frac{c^6}{c^5} = \underline{\hspace{2cm}}$$

- (b) Look at the indices in each fraction and in its answer. Describe a quicker way to find the answer without writing the powers out in full.

- (c) Use your quicker way to simplify these fractions.

$$\frac{x^{20}}{x^{13}} = \underline{\hspace{2cm}}$$

$$\frac{y^{100}}{y^{99}} = \underline{\hspace{2cm}}$$

- (d) Use your quicker way on $\frac{a^3}{a^3}$. Cancelling shows that $\frac{a^3}{a^3} = 1$. What does this suggest a^0 is equal to?

$$\frac{a^3}{a^3} = a^{\square} \quad \text{so} \quad a^0 = \underline{\hspace{2cm}}$$

Part 5: Extension: factorials, negative numbers and the fourth dimension

The **factorial** of a whole number is written with an exclamation mark after it. The expression $n!$ means multiply together every whole number from n down to 1, so $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

$$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3\,628\,800.$$

Q10. Work out each factorial.

(a) $3! =$ _____ (b) $4! =$ _____ (c) $6! =$ _____ (d) $1! =$ _____

Worked example: simplify $\frac{6!}{4!}$

Both factorials are written out in full. Every whole number from 4 downwards appears on the top and on the bottom, so all of those factors cancel.

$$\frac{6!}{4!} = \frac{6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 6 \times 5 = 30$$

Q11. Simplify each expression by writing the factorials out in full and cancelling. Do not work the factorials out first.

(a) $\frac{5!}{3!}$

(c) $\frac{8!}{6!}$

(e) $\frac{100!}{98!}$

(b) $\frac{10!}{9!}$

(d) $\frac{7!}{5! \times 2!}$

(f) $\frac{12!}{10! \times 2!}$

Q12. For any whole number n that is greater than 1, the expression $\frac{n!}{(n-1)!}$ always simplifies to n . Explain why.

Q13. When the base is negative, the brackets show that the whole negative number is multiplied by itself, so $(-2)^3 = (-2) \times (-2) \times (-2)$.

(a) Complete the table.

Expression	Written out in full	Answer
$(-2)^2$	$(-2) \times (-2)$	4
$(-2)^3$	$(-2) \times (-2) \times (-2)$	
$(-2)^4$		
$(-3)^2$		
$(-3)^3$		

(b) Write down what you notice about the sign of the answer when the index is even, and when the index is odd.

(c) Explain why this happens.

Q14. (a) Work out 3×3 and $(-3) \times (-3)$. Write down both of the numbers that give 9 when they are multiplied by themselves.

_____ and _____

The symbol $\sqrt{\quad}$ means the positive square root only, so $\sqrt{9} = 3$.

(b) Work out $\sqrt{-9}$ on your calculator. Write down what the calculator shows, and explain why.

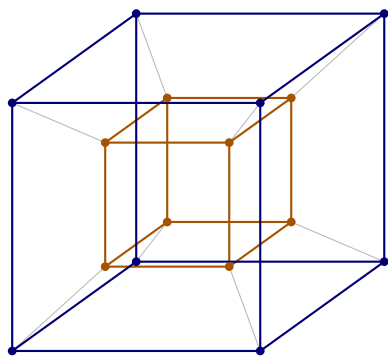
(c) Work out $(-2) \times (-2) \times (-2)$. Use your answer to write down $\sqrt[3]{-8}$.

_____ so $\sqrt[3]{-8} =$ _____

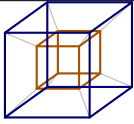
(d) Sam says that a root of a negative number can never be found. Use your answers to parts (b) and (c) to say whether Sam is right, and explain your answer.

Q15. A square is the two-dimensional shape made by sliding a line at right angles to itself, a cube is the three-dimensional shape made by sliding a square at right angles to itself. A **tesseract** is the shape made by sliding a cube at right angles to itself, in the fourth direction. There is no fourth direction in the space we live in, so a tesseract cannot be built, but it can still be drawn and its measurements can still be worked out.

Below is a drawing of tesseract, or the best we can do in three dimensions. The blue and orange corners would in fact all be the outside of the tesseract, and the fourth lines connecting the inner and outer cubes would form right angles to the remaining lines (we don't have enough dimensions to imagine this!)



(a) Complete the table. Count the corners on each picture.

Shape	point	line	square	cube	tesseract
Picture	.	—			
Dimensions	0	1	2	3	4
Corners	1	2	4		

(b) Write the number of corners of the square, the cube and the tesseract as a power of 2.

Square: _____ Cube: _____ Tesseract: _____

(c) The shape after the tesseract has five dimensions. Use the pattern in the table to write the number of corners it has as a power of 2, and then work it out.

Corners = _____ = _____

(d) One of the shapes in this pattern has 1024 corners. How many dimensions does it have? Show how you know.

- (e) A cube with edges of length 3 cm has a volume of $3^3 = 27 \text{ cm}^3$. A tesseract with edges of length 3 cm has a **hypervolume** of 3^4 cm^4 . Work out 3^4 .

$$3^4 = \underline{\hspace{4cm}} \text{ cm}^4$$

- (f) A tesseract has a hypervolume of 16 cm^4 . The length of one of its edges is the **fourth root** of 16, written $\sqrt[4]{16}$. Work out $\sqrt[4]{16}$ by finding the number that gives 16 when four of them are multiplied together.

$$\sqrt[4]{16} = \underline{\hspace{4cm}} \text{ cm}$$

- (g) Show that $(-2)^4 = 16$ as well. Explain why the edge of the tesseract must still be 2 cm.

- (h) A second tesseract has a hypervolume of 81 cm^4 . Work out the length of one of its edges.

$$\sqrt[4]{81} = \underline{\hspace{4cm}} \text{ cm}$$

- (i) A square has 4 edges, a cube has 12 edges and a tesseract has 32 edges. The 12 edges of a cube can be written as 3×2^2 . Write the edges of a square and the edges of a tesseract in the same way, as a number multiplied by a power of 2. Then use the pattern to predict how many edges the five-dimensional shape has.

$$\text{Square: } 4 = \underline{\hspace{2cm}} \quad \text{Cube: } 12 = 3 \times 2^2 \quad \text{Tesseract: } 32 = \underline{\hspace{2cm}}$$

- (j) Use both patterns to find the number of corners and the number of edges of the six-dimensional shape.

$$\text{Corners} = \underline{\hspace{4cm}} \quad \text{Edges} = \underline{\hspace{4cm}}$$

Answers

Q1. 1-D (25) 2-H (10) 3-B (8) 4-F (0.04)
5-A (0.5) 6-E (1000) 7-C (2) 8-G (2.25)

Q2. (a) $6 \times 6 = 36 \text{ m}^2$ (b) $\sqrt{144} = 12 \text{ m}$ (c) $4 \times 4 \times 4 = 64 \text{ cm}^3$ (d) $\sqrt[3]{125} = 5 \text{ cm}$ (e) $\sqrt{0.25} = 0.5 \text{ m}$ (f) Between 7 m and 8 m, because $7^2 = 49$ and $8^2 = 64$, and 50 is between 49 and 64.

Q3. (a) a^4 (b) $\frac{c^3}{d^2}$ (c) b^5 (d) $3m^2$ (e) $\frac{p^3}{q^4}$
(f) $\frac{2x^3}{y^2}$ (g) $\frac{a^2b^3}{c^2}$ (h) $\frac{t^2}{5u^3}$

Q4. (b) $y \times y \times y \times y \times y$ (c) $4 \times t \times t$ (d) $\frac{m \times m}{n \times n \times n}$
(e) $\frac{a \times a \times a \times a}{b}$

Q5. (a) $\frac{2}{3}$ (b) $\frac{5 \times 3}{5 \times 5} = \frac{3}{5}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$ (e) $\frac{3}{4}$
(f) $\frac{2}{3}$ (g) $\frac{12}{28} = \frac{3}{7}$ (h) $\frac{45}{120} = \frac{3}{8}$

Q6. (a) a^2 (b) b^3 (c) $2m$ (d) $3p^2$ (e) $\frac{x}{y}$ (f) $\frac{2t}{3}$ (g) $\frac{2x}{3y}$ (h) $\frac{5b}{a}$ (i) Two a s cancel from the top and the bottom, which leaves 1 on the top and one a on the bottom. Lena has put the a that is left on the top. The answer is $\frac{1}{a}$.

Q7. 1-D (a^3) 2-F (1) 3-H ($3a^2$) 4-G (a^2)
5-E ($\frac{a}{2}$) 6-A (b^2) 7-C (2) 8-B (a^4)

Q8. (a) There are many answers, for example $\frac{2a^3}{a}$, $\frac{4a^3}{2a}$ and $\frac{6a^5}{3a^3}$. (b) There are many answers, for example $\frac{a^3}{ab}$ or $\frac{a^2b}{b^2}$. (c) There are many answers, for example $\frac{z}{xy^2z}$ or $\frac{xz}{x^2y^2z}$. (d) Cancelling divides the top and the bottom by the same factor, and a number divided by itself is 1, not 0. What is left on the top is 1 and what is left on the bottom is 1, so $\frac{3a^2}{3a^2} = \frac{1}{1} = 1$.

Q9. (a) a^4 , b^5 , c (b) Subtract the index on the bottom from the index on the top, and keep the same base. (c) x^7 , y (d) $\frac{a^3}{a^3} = a^{3-3} = a^0$, and cancelling gives 1, so $a^0 = 1$.

Q10. (a) 6 (b) 24 (c) 720 (d) 1

Q11. (a) $5 \times 4 = 20$ (b) 10 (c) $8 \times 7 = 56$ (d) $\frac{7 \times 6}{2 \times 1} = 21$ (e) $100 \times 99 = 9900$ (f) $\frac{12 \times 11}{2 \times 1} = 66$

Q12. $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$ and $(n-1)! = (n-1) \times (n-2) \times \cdots \times 1$. Every factor of $(n-1)!$ also appears in $n!$, so all of them cancel. The only factor left on the top is n , and the bottom is 1.

Q13. (a) $(-2)^3 = -8$; $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$; $(-3)^2 = (-3) \times (-3) = 9$; $(-3)^3 = (-3) \times (-3) \times (-3) = -27$. (b) An even index gives a positive answer and an odd index gives a negative answer. (c) Two negative numbers multiplied together give a positive answer, so the negative signs pair up. An even index leaves no negative sign over, and an odd index leaves one negative sign over.

Q14. (a) $3 \times 3 = 9$ and $(-3) \times (-3) = 9$, so the two numbers are 3 and -3. (b) The calculator shows an error. No number multiplied by itself gives a negative answer, because a positive multiplied by a positive is positive, and a negative multiplied by a negative is also positive. (c) $(-2) \times (-2) \times (-2) = -8$, so $\sqrt[3]{-8} = -2$. (d) Sam is not right. A negative number has no square root, because an even index always gives a positive answer, but it does have a cube root, because an odd index can give a negative answer.

Q15. (a) The cube has 8 corners and the tesseract has 16 corners. (b) Square: $4 = 2^2$; cube: $8 = 2^3$; tesseract: $16 = 2^4$. The index is the number of dimensions. (c) $2^5 = 32$ corners (d) $1024 = 2^{10}$, so it has 10 dimensions. (e) $3^4 = 3 \times 3 \times 3 \times 3 = 81 \text{ cm}^4$ (f) $\sqrt[4]{16} = 2 \text{ cm}$, because $2 \times 2 \times 2 \times 2 = 16$. (g) $(-2) \times (-2) \times (-2) \times (-2) = 16$, because the index is even. The edge must still be 2 cm, because a length cannot be negative. (h) $\sqrt[4]{81} = 3 \text{ cm}$, because $3 \times 3 \times 3 \times 3 = 81$. (i) Square: $4 = 2 \times 2^1$; tesseract: $32 = 4 \times 2^3$. The number in front is the number of dimensions and the index is one less than it, so the five-dimensional shape has $5 \times 2^4 = 80$ edges. (j) Corners = $2^6 = 64$; edges = $6 \times 2^5 = 192$.