

Recurring Decimals into Fractions

Worked Solutions

Part 1: Reading and Writing Recurring Decimals

Q1. Write out the first eight decimal places.

Worked solution.

- (a) $0.\dot{5} = 0.55555555$
- (b) $0.1\dot{8} = 0.18181818$
- (c) $0.3\dot{7} = 0.37777777$
- (d) $0.20\dot{6} = 0.20620620$
- (e) $0.45\dot{1} = 0.45151515$
- (f) $0.\dot{9} = 0.99999999$

Q2. Write each using dot notation.

Worked solution.

- (a) $0.8888\dots = 0.\dot{8}$
- (b) $0.636363\dots = 0.\dot{6}\dot{3}$
- (c) $0.24444\dots = 0.2\dot{4}$
- (d) $0.157157\dots = 0.1\dot{5}\dot{7}$
- (e) $0.91666\dots = 0.91\dot{6}$
- (f) $0.40505\dots = 0.40\dot{5}$

Q3.

Worked solution.

- (a) We have $0.4\dot{5} = 0.454545\dots$ and $0.4\dot{5} = 0.455555\dots$

They agree in the first two decimal places. At the third decimal place, the first number has 4 and the second has 5. Hence $0.4\dot{5}$ is bigger.

- (b) There are many possible answers. For example,

$$0.\dot{3}\dot{5} = 0.353535\dots$$

is bigger than $0.\dot{3} = 0.3333\dots$ and smaller than $0.\dot{4} = 0.4444\dots$

Part 2: Finding Fractions from Recurring Decimals

Q4. Fill in the gaps to turn $0.\dot{7}$ into a fraction.

Worked solution. Step 1: Let $x = 0.\dot{7} = 0.7777\dots$

Step 2: Multiplying by 10 shifts every digit one place to the left:

$$10x = 7.7777\dots$$

Step 3: The tails are identical, so they cancel on subtraction:

$$10x - x = 7.7777\dots - 0.7777\dots = 7.$$

Therefore

$$9x = 7.$$

Step 4: Divide both sides by 9:

$$x = \frac{7}{9}.$$

Q5. Turn $0.\dot{8}$ into a fraction.

Worked solution. Let $x = 0.\dot{8} = 0.8888\dots$

$$10x = 8.8888\dots$$

Subtract:

$$10x - x = 8,$$

so

$$9x = 8.$$

Hence

$$x = \frac{8}{9}.$$

Q6. Turn each into a fraction in its simplest form.

Worked solution.

(a) $0.\dot{2}$: let $x = 0.2222\dots$. Then $10x = 2.2222\dots$, so $9x = 2$ and $x = \frac{2}{9}$.

(b) $0.\dot{5}$: let $x = 0.5555\dots$. Then $10x = 5.5555\dots$, so $9x = 5$ and $x = \frac{5}{9}$.

(c) $0.\dot{6}$: let $x = 0.6666\dots$. Then $10x = 6.6666\dots$, so $9x = 6$ and $x = \frac{6}{9} = \frac{2}{3}$.

(d) $0.\dot{3}$: let $x = 0.3333\dots$. Then $10x = 3.3333\dots$, so $9x = 3$ and $x = \frac{3}{9} = \frac{1}{3}$.

Part 3: Longer Repeating Blocks

Q7. Fill in the gaps to turn $0.\dot{7}\dot{2}$ into a fraction.

Worked solution. Step 1: Let $x = 0.\dot{7}\dot{2} = 0.727272\dots$

Step 2: The repeating block has 2 digits, so multiply by 100:

$$100x = 72.727272 \dots$$

Step 3: The tails now match, so

$$100x - x = 72.727272 \dots - 0.727272 \dots = 72.$$

Thus

$$99x = 72.$$

Step 4: Divide both sides by 99:

$$x = \frac{72}{99} = \frac{8}{11}.$$

Q8. Turn these into fractions in their simplest form.

Worked solution.

(a) $0.\dot{1}\dot{5}$: let $x = 0.151515 \dots$. Then $100x = 15.151515 \dots$, so $99x = 15$ and

$$x = \frac{15}{99} = \frac{5}{33}.$$

(b) $0.0\dot{6}$: let $x = 0.060606 \dots$. Then $100x = 6.060606 \dots$, so $99x = 6$ and

$$x = \frac{6}{99} = \frac{2}{33}.$$

(c) $0.\dot{5}\dot{4}$: let $x = 0.545454 \dots$. Then $100x = 54.545454 \dots$, so $99x = 54$ and

$$x = \frac{54}{99} = \frac{6}{11}.$$

(d) $0.9\dot{0}$: let $x = 0.909090 \dots$. Then $100x = 90.909090 \dots$, so $99x = 90$ and

$$x = \frac{90}{99} = \frac{10}{11}.$$

(e) $0.\dot{1}0\dot{8}$: the repeating block is 108, which has 3 digits, so multiply by 1000. Let $x = 0.108108 \dots$. Then $1000x = 108.108108 \dots$, so $999x = 108$ and

$$x = \frac{108}{999} = \frac{4}{37}.$$

Q9.

(a) To decide what to multiply by, count the number of digits in the repeating block. If the block has n digits, multiply by 10^n . This moves the decimal point n places, so the repeating block lines up exactly with the block in the original decimal. The tails then cancel when the two decimals are subtracted.

(b) $0.\dot{1}\dot{2} = 0.121212 \dots$. Let $x = 0.\dot{1}\dot{2}$. Then $100x = 12.121212 \dots$, so

$$99x = 12 \quad \text{and} \quad x = \frac{12}{99} = \frac{4}{33}.$$

Ben has simply simplified Kayla's fraction. Both are correct.

Part 4: When the Repeating Part Does Not Start Straight Away

Q10. Fill in the gaps to turn $0.2\dot{7}$ into a fraction.

Worked solution. Let $x = 0.2\dot{7} = 0.27777\dots$

Shift past the 2:

$$10x = 2.7777\dots$$

Shift one more place:

$$100x = 27.7777\dots$$

Subtract the first from the second:

$$100x - 10x = 27.7777\dots - 2.7777\dots = 25.$$

Thus

$$90x = 25$$

and

$$x = \frac{25}{90} = \frac{5}{18}.$$

Q11. Turn these into fractions in their simplest form.

Worked solution.

(a) $0.4\dot{3} = 0.4333\dots$ Let $x = 0.4333\dots$

$$10x = 4.3333\dots, \quad 100x = 43.3333\dots$$

Subtract:

$$90x = 39$$

so

$$x = \frac{39}{90} = \frac{13}{30}.$$

(b) $0.58\dot{3} = 0.58333\dots$ Let $x = 0.58333\dots$

$$100x = 58.3333\dots, \quad 1000x = 583.3333\dots$$

Subtract:

$$900x = 525$$

so

$$x = \frac{525}{900} = \frac{7}{12}.$$

(c) $0.08\dot{3} = 0.08333\dots$ Let $x = 0.08333\dots$

$$100x = 8.3333\dots, \quad 1000x = 83.3333\dots$$

Subtract:

$$900x = 75$$

so

$$x = \frac{75}{900} = \frac{1}{12}.$$

(d) $0.1\dot{2}\dot{4} = 0.1242424\dots$. Let $x = 0.1242424\dots$

$$10x = 1.242424\dots, \quad 1000x = 124.242424\dots$$

Subtract:

$$990x = 124 - 1 = 123$$

so

$$x = \frac{123}{990} = \frac{41}{330}.$$

Q12. Turn these into improper fractions in their simplest form.

Worked solution.

(a) $2.\dot{7} = 2.7777\dots$. Let $x = 2.7777\dots$

$$10x = 27.7777\dots$$

Subtract:

$$9x = 25$$

so

$$x = \frac{25}{9}.$$

(b) $1.4\dot{5} = 1.454545\dots$. Let $x = 1.454545\dots$

$$100x = 145.454545\dots$$

Subtract:

$$99x = 144$$

so

$$x = \frac{144}{99} = \frac{16}{11}.$$

(c) $3.\dot{1}\dot{8} = 3.181818\dots$. Let $x = 3.181818\dots$

$$100x = 318.181818\dots$$

Subtract:

$$99x = 315$$

so

$$x = \frac{315}{99} = \frac{35}{11}.$$

Q13. Mixed practice.

Worked solution.

(a) $0.\dot{8}$: $x = 0.8888\dots$, so $10x = 8.8888\dots$. Therefore $9x = 8$ and $x = \frac{8}{9}$.

(b) $0.\dot{6}\dot{3}$: $x = 0.636363\dots$, so $100x = 63.636363\dots$. Therefore $99x = 63$ and $x = \frac{63}{99} = \frac{7}{11}$.

(c) $0.2\dot{5}$: $x = 0.2555\dots$, so $10x = 2.5555\dots$ and $100x = 25.5555\dots$. Subtracting gives $90x = 23$, so $x = \frac{23}{90}$.

(d) $0.\dot{0}\dot{2}\dot{7}$: $x = 0.027027\dots$, so $1000x = 27.027027\dots$. Therefore $999x = 27$ and $x = \frac{27}{999} = \frac{1}{37}$.

- (e) $0.7\dot{2}$: $x = 0.7222\dots$, so $10x = 7.2222\dots$ and $100x = 72.2222\dots$. Subtracting gives $90x = 65$, so $x = \frac{65}{90} = \frac{13}{18}$.
- (f) $0.1\dot{2}$: $x = 0.121212\dots$, so $100x = 12.121212\dots$. Therefore $99x = 12$ and $x = \frac{12}{99} = \frac{4}{33}$.
- (g) $0.41\dot{6}$: $x = 0.41666\dots$, so $100x = 41.6666\dots$ and $1000x = 416.6666\dots$. Subtracting gives $900x = 375$, so $x = \frac{375}{900} = \frac{5}{12}$.
- (h) $0.06\dot{4}$: $x = 0.06444\dots$, so $100x = 6.4444\dots$ and $1000x = 64.4444\dots$. Subtracting gives $900x = 58$, so $x = \frac{58}{900} = \frac{29}{450}$.

Part 5: Challenges

Q14.

- (a) Let $x = 0.\dot{9} = 0.9999\dots$

$$10x = 9.9999\dots$$

Subtract:

$$10x - x = 9$$

so

$$9x = 9$$

and hence

$$x = 1.$$

- (b) A good reply to Ella is:

The algebra is not a trick. If $0.\dot{9} = x$, then $10x = 9.\dot{9}$, so subtracting gives $9x = 9$, and therefore $x = 1$.

Also, $\frac{1}{3} = 0.\dot{3}$. Multiplying both sides by 3 gives

$$1 = 0.\dot{9}.$$

On the number line, the numbers $0.9, 0.99, 0.999, \dots$ get closer and closer to 1. Each new one closes half of the remaining gap. Since there is no positive gap left, $0.\dot{9}$ must actually equal 1.

- (c) There is no largest number smaller than 1. If someone names a number $a < 1$, then

$$a < \frac{a+1}{2} < 1.$$

So there is always another number between a and 1, which is bigger than a but still smaller than 1. This is also why $0.\dot{9}$ cannot be “just below” 1: there is no room for a largest number below 1.

Q15.

Worked solution.

(a) Using short division:

$$\begin{aligned}\frac{1}{2} &= 0.5, & \frac{1}{3} &= 0.\dot{3}, & \frac{1}{4} &= 0.25, & \frac{1}{5} &= 0.2, \\ \frac{1}{6} &= 0.1\dot{6}, & \frac{1}{7} &= 0.\dot{1}4285\dot{7}, & \frac{1}{8} &= 0.125, \\ \frac{1}{9} &= 0.\dot{1}, & \frac{1}{10} &= 0.1, & \frac{1}{11} &= 0.\dot{0}\dot{9}, & \frac{1}{12} &= 0.08\dot{3}, & \frac{1}{16} &= 0.0625.\end{aligned}$$

The terminating decimals are

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}.$$

(b) The terminating denominators are:

$$2 = 2, \quad 4 = 2^2, \quad 5 = 5, \quad 8 = 2^3, \quad 10 = 2 \times 5, \quad 16 = 2^4.$$

They all have only the prime factors 2 and 5.

(c)

$$\begin{aligned}\frac{1}{15} &\text{ recurs, since } 15 = 3 \times 5. \\ \frac{1}{40} &\text{ terminates, since } 40 = 2^3 \times 5. \\ \frac{1}{35} &\text{ recurs, since } 35 = 5 \times 7. \\ \frac{1}{64} &\text{ terminates, since } 64 = 2^6.\end{aligned}$$

(d) A terminating decimal can be written with denominator

$$10, 100, 1000, \dots$$

which are all of the form $2^k 5^k$. So after a fraction is written in lowest terms, its denominator can contain only 2s and 5s.

Conversely, if the denominator contains only 2s and 5s, we can multiply the numerator and denominator by enough extra 2s or 5s to make the denominator a power of 10, and then the decimal stops. Any other prime factor can never be removed, so such a fraction recurs.

Q16.

Worked solution.

(a)

$$\begin{aligned}\frac{2}{7} &= 0.\dot{2}8571\dot{4}, \\ \frac{3}{7} &= 0.\dot{4}2857\dot{1}, \\ \frac{4}{7} &= 0.\dot{5}7142\dot{8}, \\ \frac{5}{7} &= 0.\dot{7}1428\dot{5}, \\ \frac{6}{7} &= 0.\dot{8}5714\dot{2}.\end{aligned}$$

- (b) Every seventh uses the same six digits in the same cyclic order:

$$1 \rightarrow 4 \rightarrow 2 \rightarrow 8 \rightarrow 5 \rightarrow 7 \rightarrow 1.$$

Each fraction simply starts at a different point in that cycle.

- (c) Let $x = 0.\dot{2}85714 = 0.285714285714\dots$

The repeating block has 6 digits, so multiply by 1 000 000:

$$1\,000\,000x = 285\,714.285714285714\dots$$

Subtract x :

$$999\,999x = 285\,714.$$

Therefore

$$x = \frac{285\,714}{999\,999}.$$

Dividing the numerator and denominator by 142 857 gives

$$x = \frac{2}{7},$$

which agrees with part (a).

Q17.

Worked solution.

- (a) One valid example is $0.\dot{2}7$. Let $x = 0.\dot{2}7 = 0.272727\dots$

$$100x = 27.272727\dots$$

Subtract:

$$99x = 27,$$

so

$$x = \frac{27}{99} = \frac{3}{11}.$$

So $0.\dot{2}7$ is a recurring decimal with denominator 11.

- (b) First,

$$\frac{5}{6} = 0.8333\dots = 0.8\dot{3}.$$

Check by turning it back: let $x = 0.8\dot{3} = 0.8333\dots$

$$10x = 8.3333\dots, \quad 100x = 83.3333\dots$$

Subtract:

$$100x - 10x = 83.3333\dots - 8.3333\dots = 75.$$

Thus

$$90x = 75,$$

so

$$x = \frac{75}{90} = \frac{5}{6}.$$

- (c) Yes. Suppose the repeating block has n digits and write the block as the whole number N . Shifting the decimal point by n places gives

$$10^n x = N + x,$$

because the same repeating tail appears again. Therefore

$$(10^n - 1)x = N,$$

so

$$x = \frac{N}{10^n - 1},$$

which is a fraction.

If the decimal has some non-repeating digits at the front, first multiply by a power of 10 to move past those digits. Then apply the same idea to the remaining recurring part. So the subtraction always works.

- (d) By part (c), every recurring decimal is equal to a fraction. But π is not equal to a fraction; it is irrational. Therefore no recurring decimal can be exactly equal to π . Its decimal expansion goes on forever but never settles into a repeating block.