

Recurring Decimals into Fractions

Every recurring decimal — however long and however complicated — can be written as a fraction. Follow the examples in this sheet to learn how to convert recurring decimals to fractions.

Part 1: Reading and writing recurring decimals

When you divide one whole number by another, the digits after the point either stop, or fall into a block that repeats forever. Repeating blocks are marked with a dot over the **first** digit and a dot over the **last** digit.

the block that repeats

$$0 \cdot \overbrace{27} \text{ } 27 \overbrace{27} \text{ } 27 \cdots \quad \text{so we write it } 0.\dot{2}\dot{7}$$

So $0.\dot{6} = 0.666666\dots$, $0.\dot{2}\dot{7} = 0.272727\dots$, $0.\dot{4}0\dot{9} = 0.409409\dots$ and $0.1\dot{6} = 0.166666\dots$ — notice that in the last one only the 6 carries a dot, so only the 6 repeats. The 1 only is found in the tenths column, and never comes back.

Q1. Write out the first eight decimal places of each number.

- (a) $0.\dot{5} = 0.\underline{\hspace{2cm}}$ (c) $0.3\dot{7} = 0.\underline{\hspace{2cm}}$ (e) $0.4\dot{5}\dot{1} = 0.\underline{\hspace{2cm}}$
(b) $0.1\dot{8} = 0.\underline{\hspace{2cm}}$ (d) $0.\dot{2}0\dot{6} = 0.\underline{\hspace{2cm}}$ (f) $0.\dot{9} = 0.\underline{\hspace{2cm}}$

Q2. Write each of these using dot notation.

- (a) $0.8888\dots = \underline{\hspace{2cm}}$ (c) $0.24444\dots = \underline{\hspace{2cm}}$ (e) $0.91666\dots = \underline{\hspace{2cm}}$
(b) $0.636363\dots = \underline{\hspace{2cm}}$ (d) $0.157157\dots = \underline{\hspace{2cm}}$ (f) $0.40505\dots = \underline{\hspace{2cm}}$

Q3. (a) Which is bigger, $0.\dot{4}\dot{5}$ or $0.4\dot{5}$? Write both out to six decimal places to help you decide, and explain your answer.

- (b) Write down a recurring decimal that lies between $0.\dot{3}$ and $0.\dot{4}$. $\underline{\hspace{2cm}}$

Part 2: Finding fractions from recurring decimals

The trick is to find two numbers whose recurring tails are **exactly the same**, and then subtract one from the other. The tails cancel, and what is left has no decimals in it at all.

To do this we have to use an **algebraic variable**, which we will call x . x stands for the recurring decimal we are trying to convert to a fraction.

Worked example: turn $0.\dot{4}$ into a fraction

Step 1 — introduce x We write $x = 0.\dot{4}$, so $x = 0.4444\dots$

Step 2 — shift the place values. Multiplying by 10 moves every digit one place to the left:
 $10x = 4.4444\dots$ (remember $10x = 10 \times x$).

Step 3 — subtract. The two tails are identical, so they cancel out when we subtract: $10x - x$.

$$\begin{array}{rcl} 10x & = & 4 \cdot \text{4 4 4 4 4} \dots \\ x & = & 0 \cdot \text{4 4 4 4 4} \dots \\ \hline 9x & = & 4 \cdot 0 0 0 0 0 \dots \end{array} \quad \begin{array}{l} \text{the shaded tails are the same digits in the} \\ \text{same columns, so they subtract to nothing} \end{array}$$

So $10x - x = 4$, which means $9x = 4$.

Step 4 — solve. We have 9 lots of x , so we divide both sides by 9:

$$\begin{array}{rcl} 9x & = & 4 \\ \div 9 \quad \curvearrowright & & \quad \curvearrowright \div 9 \\ x & = & \frac{4}{9} \end{array}$$

Q4. Fill in the gaps to turn $0.\dot{7}$ into a fraction.

Step 1 — introduce x We write $x = 0.\dot{7}$, so $x = \underline{\hspace{2cm}}$

Step 2 — shift the place values. Multiplying by 10 moves every digit one place to the left:
 $10x = \underline{\hspace{2cm}}$ (remember $10x = 10 \times x$).

Step 3 — subtract. The two tails are identical, so they cancel out when we subtract: $10x - x$.

$$\begin{array}{rcl} 10x & = & \text{—} \cdot \text{ } \dots \\ x & = & 0 \cdot \text{ } \dots \\ \hline 9x & = & \text{—} \cdot \text{ } \dots \end{array} \quad \begin{array}{l} \text{the shaded tails are the same digits in the} \\ \text{same columns, so they subtract to nothing} \end{array}$$

So $10x - x = \underline{\hspace{2cm}}$, which means $9x = \underline{\hspace{2cm}}$.

Step 4 — solve. We have $\underline{\hspace{1cm}}$ lots of x , so we divide both sides by $\underline{\hspace{1cm}}$. Set your working out like the worked example.

Q5. Turn $0.\dot{8}$ into a fraction, using the same four steps.

Step 1 — introduce x _____

Step 2 — shift the place values. _____

Step 3 — subtract. The two tails are identical, so they cancel out when we subtract:
 $10x - x$.

$10x =$	— ·	_____	$\cdots$
$x =$	0 ·	_____	$\cdots$
$9x =$	— ·	_____	$\cdots$

the shaded tails are the same digits in the
 same columns, so they subtract to nothing

So $10x - x =$ _____, which means $9x =$ _____.

Step 4 — solve. Divide both sides by ____.

Q6. Turn each of these into a fraction in its simplest form. Write out all four steps each time.

- (a) $0.\dot{2}$
(b) $0.\dot{5}$
(c) $0.\dot{6}$
(d) $0.\dot{3}$

Part 3: Longer repeating blocks

Try $0.\dot{3}\dot{6}$ the same way and the tails do not cancel. Multiplying by 10 shifts the place values by one, which puts the two digits of the block the wrong way round. A block of *two* digits has to be shifted by *two* place values, so we multiply by 100 instead.

$$\begin{array}{rcl} 10x & = & 3 \cdot 6363\cdots \\ x & = & 0 \cdot 3636\cdots \end{array}$$

tails differ — nothing cancels

$$\begin{array}{rcl} 100x & = & 36 \cdot 3636\cdots \\ x & = & 0 \cdot 3636\cdots \end{array}$$

tails match — these cancel

Worked example: turn $0.\dot{3}\dot{6}$ into a fraction

Step 1 — introduce x

$x = 0.\dot{3}\dot{6}$, so $x = 0.363636\dots$

Step 2 — shift the place values.

$100x = 36.363636\dots$

the block is 2 digits long

Step 3 — subtract.

$100x - x = 36$

so $99x = 36$

Step 4 — solve.

$x = \frac{36}{99} = \frac{4}{11}$

Q7. Fill in the gaps to turn $0.\dot{7}\dot{2}$ into a fraction.

- Step 1 — introduce x $x = 0.727272\dots$
- Step 2 — shift the place values. The block is _____ digits long, so _____ $x =$ _____
- Step 3 — subtract._____ $x =$ _____
- Step 4 — solve. $x =$ _____

Q8. Turn these into fractions in their simplest form. Part (e) has a block of three digits, so think about what to multiply by.

- (a) $0.\dot{1}\dot{5}$
- (b) $0.\dot{0}\dot{6}$
- (c) $0.\dot{5}\dot{4}$
- (d) $0.\dot{9}\dot{0}$
- (e) $0.\dot{1}0\dot{8}$

- Q9.** (a) Explain, in your own words, how you decide what to multiply x by.
- (b) Kayla converts $0.\dot{1}\dot{2}$ and gets $\frac{12}{99}$. Ben converts the same number and gets $\frac{4}{33}$. Who is right?

Part 4: When the repeating part does not start straight away

$0.1\dot{6}$ is harder, because the 1 is in the tenths column and does not repeat. Multiplying by 10 gives a number whose tail is all sixes, but that tail does not match the tail of x itself. Instead we shift the place values *twice*, and subtract the two results from each other.

$x =$	$0 \cdot 1\ 6\ 6\ 6\ 6\ldots$	the tail of x starts with a 1, so it will
$10x =$	$1 \cdot 6\ 6\ 6\ 6\ 6\ldots$	never match
$100x =$	$16 \cdot 6\ 6\ 6\ 6\ 6\ldots$	these two match, so subtract these two

Worked example: turn $0.1\dot{6}$ into a fraction

Step 1 — introduce x	$x = 0.1\dot{6}$, so $x = 0.16666\ldots$	
Step 2 — shift the place values.	$10x = 1.6666\ldots$	shift past the 1
	$100x = 16.6666\ldots$	shift one more place
Step 3 — subtract.	$100x - 10x = 16 - 1 = 15$	so $90x = 15$
Step 4 — solve.	$x = \frac{15}{90} = \frac{1}{6}$	

Q10. Fill in the gaps to turn $0.2\dot{7}$ into a fraction.

Step 1 — introduce x	$x = 0.27777\ldots$
Step 2 — shift the place values. Shift past the 2:	$10x = \underline{\hspace{2cm}}$
Shift one more place:	$100x = \underline{\hspace{2cm}}$
Step 3 — subtract.	$100x - 10x = \underline{\hspace{2cm}}$
so	$90x = \underline{\hspace{2cm}}$
Step 4 — solve.	$x = \underline{\hspace{2cm}}$

Q11. Turn these into fractions in their simplest form. In (d) the repeating block is two digits long, so think carefully about which two multiples of x you need.

(a) $0.4\dot{3}$

(b) $0.58\dot{3}$

(c) $0.08\dot{3}$

(d) $0.1\dot{2}4$

Q12. The same four steps work when the number is bigger than 1. Turn these into improper fractions in their simplest form.

(a) $2.\dot{7}$

(b) $1.4\dot{5}$

(c) $3.\dot{1}8$

Q13. Mixed practice. Decide for yourself what to multiply by each time. Simplify every answer.

(a) $0.\dot{8}$

(c) $0.2\dot{5}$

(e) $0.7\dot{2}$

(g) $0.41\dot{6}$

(b) $0.\dot{6}\dot{3}$

(d) $0.\dot{0}2\dot{7}$

(f) $0.\dot{1}\dot{2}$

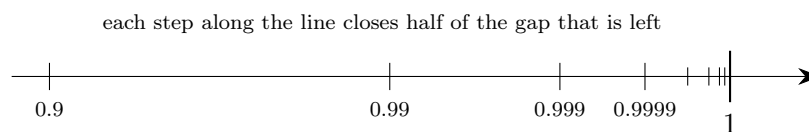
(h) $0.06\dot{4}$

Part 5: Challenges

Q14. The value of $0.\dot{9}$.

(a) Use the four steps from Part 2 on $x = 0.\dot{9}$. What fraction do you get?

(b) Ella says: “That has to be wrong. $0.9999\dots$ is obviously a bit less than 1.” The number line below marks 0.9, then 0.99, then 0.999, and so on. Use it, together with the fact that $\frac{1}{3} = 0.\dot{3}$, to write a reply to Ella.



(c) What is the largest number that is smaller than 1? Explain.

Q15. Which fractions recur? Some fractions give decimals that stop — these are called *terminating* decimals — and some give decimals that recur forever.

(a) Complete the table using short division. Circle the terminating ones.

$\frac{1}{2} = 0.5$	$\frac{1}{3} =$	$\frac{1}{4} =$	$\frac{1}{5} =$	$\frac{1}{6} =$	$\frac{1}{7} =$
$\frac{1}{8} =$	$\frac{1}{9} =$	$\frac{1}{10} =$	$\frac{1}{11} =$	$\frac{1}{12} =$	$\frac{1}{16} =$

(b) Write each terminating denominator as a product of its prime factors. What do they all have in common?

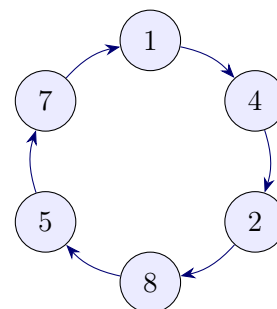
(c) Without dividing, predict whether each of these terminates or recurs, and say how you know.

$$\frac{1}{15} \quad \frac{1}{40} \quad \frac{1}{35} \quad \frac{1}{64}$$

(d) Why does having only 2s and 5s in the denominator make the decimal stop? *Hint: what is special about 2×5 ?*

Q16. Sevenths. One seventh has a repeating block of six digits: $\frac{1}{7} = 0.\dot{1}4285\dot{7}$.

- (a) Use short division to work out $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$ as recurring decimals.



reading round the wheel
from a different starting
digit gives a different seventh

- (b) What do you notice? Use the wheel to describe what happens.

- (c) Use the four steps on $0.28571\dot{4}$ to turn it back into a fraction, and check that it agrees with your answer to part (a).

Q17. Make your own.

- (a) Invent a recurring decimal that converts to a fraction with denominator 11. Show that it works.
- (b) Find the recurring decimal that is exactly $\frac{5}{6}$, then use the four steps to turn it back into $\frac{5}{6}$.
- (c) Is *every* recurring decimal equal to a fraction? Explain, in your own words, why the subtraction always works, whatever recurring decimal you start with.
- (d) Ben says: “ $\pi = 3.14159\dots$ goes on forever, so there must be a recurring decimal that is exactly equal to π .” Use your answer to part (c) to explain why Ben is wrong.

Answers

Q1. (a) 0.55555555 (b) 0.18181818 (c) 0.37777777 (d) 0.20620620 (e) 0.45151515 (f) 0.99999999

Q2. (a) $0.\dot{8}$ (b) $0.\dot{6}\dot{3}$ (c) $0.2\dot{4}$ (d) $0.1\dot{5}\dot{7}$ (e) $0.91\dot{6}$ (f) $0.40\dot{5}$

Q3. (a) $0.4\dot{5} = 0.454545$ and $0.4\dot{5} = 0.455555$, so $0.4\dot{5}$ is bigger — they agree to two decimal places, but the third digit is 5 against 4. (b) Many answers, for example $0.3\dot{5} = 0.353535\dots$

Q4. $10x = 7.7777\dots$; $10x - x = 7$; $9x = 7$; $x = \frac{7}{9}$

Q5. $x = 0.8888\dots$; $10x = 8.8888\dots$; $10x - x = 8$, so $9x = 8$; $x = \frac{8}{9}$

Q6. (a) $\frac{2}{9}$ (b) $\frac{5}{9}$ (c) $\frac{6}{9} = \frac{2}{3}$ (d) $\frac{3}{9} = \frac{1}{3}$

Q7. The block is 2 digits long, so multiply by 100: $100x = 72.727272\dots$; $99x = 72$; $x = \frac{72}{99} = \frac{8}{11}$

Q8. (a) $\frac{15}{99} = \frac{5}{33}$ (b) $\frac{6}{99} = \frac{2}{33}$ (c) $\frac{54}{99} = \frac{6}{11}$ (d) $\frac{90}{99} = \frac{10}{11}$ (e) $\frac{108}{999} = \frac{4}{37}$, multiplying by 1000

Q9. (a) Count the digits in the repeating block. If there are n of them, multiply by 1 followed by n zeros, so that the block shifts a whole number of times and the tails line up again. (b) Both are right: $\frac{12}{99} = \frac{4}{33}$. Ben has simplified his, which is what the question asks for.

Q10. $10x = 2.7777\dots$; $100x = 27.7777\dots$; $100x - 10x = 27 - 2 = 25$; $90x = 25$; $x = \frac{25}{90} = \frac{5}{18}$

Q11. (a) $\frac{39}{90} = \frac{13}{30}$ (b) $\frac{525}{900} = \frac{7}{12}$ (c) $\frac{75}{900} = \frac{1}{12}$ (d) $\frac{123}{990} = \frac{41}{330}$, using $1000x - 10x$

Q12. (a) $\frac{25}{9}$ (b) $\frac{144}{99} = \frac{16}{11}$ (c) $\frac{315}{99} = \frac{35}{11}$

Q13. (a) $\frac{8}{9}$ (b) $\frac{63}{99} = \frac{7}{11}$ (c) $\frac{23}{90}$ (d) $\frac{27}{999} = \frac{1}{37}$ (e) $\frac{65}{90} = \frac{13}{18}$ (f) $\frac{12}{99} = \frac{4}{33}$ (g) $\frac{375}{900} = \frac{5}{12}$ (h) $\frac{58}{900} = \frac{29}{450}$

Q14. (a) $10x = 9.9999\dots$, so $9x = 9$ and $x = \frac{9}{9} = 1$. (b) The method is sound, so $0.\dot{9}$ really is 1. Every step along the number line closes half of what is left, so the marks get as close to 1 as you like — there is no room left between $0.\dot{9}$ and 1 for them to be different numbers. Also $\frac{1}{3} = 0.\dot{3}$, and multiplying both sides

by 3 gives $1 = 0.\dot{9}$. (c) There isn't one. Whatever number you name below 1, there is always another number between it and 1 — which is exactly why $0.\dot{9}$ has to be 1 rather than sit just below it.

Q15. (a) $\frac{1}{3} = 0.\dot{3}$, $\frac{1}{4} = 0.25$, $\frac{1}{5} = 0.2$, $\frac{1}{6} = 0.1\dot{6}$, $\frac{1}{7} = 0.14285\dot{7}$, $\frac{1}{8} = 0.125$, $\frac{1}{9} = 0.\dot{1}$, $\frac{1}{10} = 0.1$, $\frac{1}{11} = 0.0\dot{9}$, $\frac{1}{12} = 0.08\dot{3}$, $\frac{1}{16} = 0.0625$. The terminating ones are $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}$. (b) $2, 2^2, 5, 2^3, 2 \times 5, 2^4$ — every one is built out of 2s and 5s only. (c) $\frac{1}{15}$ recurs, since $15 = 3 \times 5$ and the 3 is not allowed; $\frac{1}{40}$ terminates, since $40 = 2^3 \times 5$; $\frac{1}{35}$ recurs, since $35 = 5 \times 7$; $\frac{1}{64}$ terminates, since $64 = 2^6$. (d) A terminating decimal is just a fraction over 10, 100, 1000, $\dots$, and $10 = 2 \times 5$. So a fraction can be rewritten over a power of ten exactly when its denominator is built from 2s and 5s; any other prime factor can never be cancelled away.

Q16. (a) $\frac{2}{7} = 0.28571\dot{4}$, $\frac{3}{7} = 0.42857\dot{1}$, $\frac{4}{7} = 0.57142\dot{8}$, $\frac{5}{7} = 0.71428\dot{5}$, $\frac{6}{7} = 0.85714\dot{2}$. (b) They all use the same six digits in the same cyclic order, $1 \rightarrow 4 \rightarrow 2 \rightarrow 8 \rightarrow 5 \rightarrow 7 \rightarrow 1$; each seventh simply starts at a different place on the wheel. (c) The block is 6 digits long, so multiply by 1 000 000: this gives $999\,999x = 285\,714$, so $x = \frac{285714}{999999} = \frac{2}{7}$, dividing top and bottom by 142 857.

Q17. (a) Many answers. For example $0.2\dot{7}$ gives $99x = 27$, so $x = \frac{27}{99} = \frac{3}{11}$; any two-digit block that is a multiple of 9 but not of 11 will do. (b) $\frac{5}{6} = 0.8\dot{3}$, and checking: $100x - 10x = 83 - 8 = 75$, so $90x = 75$ and $x = \frac{75}{90} = \frac{5}{6}$. (c) Yes. Whatever the decimal, shifting it by the length of the repeating block always produces a second number with an identical tail, so the subtraction always leaves a whole number. That gives an equation of the form (whole number) $\times x =$ (whole number), and dividing gives a fraction. (d) π is not a fraction, and part (c) shows that every recurring decimal is one. So no recurring decimal can equal π : its digits go on forever without ever settling into a repeating block.